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Semigroups with inverse skeletons and Zappa-Szép products

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Abstract. The aim of this paper is to study semigroups possessing E-regular elements, where an element a of a semigroup S is *E-regular* if a has an inverse a° such that $aa^\circ, a^\circ a$ lie in $E \subseteq E(S)$. Where S possesses 'enough' (in a precisely defined way) E-regular elements, analogues of Green's lemmas and even of Green's theorem hold, where Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}$ and $\mathcal{D}$ are replaced by $\widetilde{\mathcal{R}}_E, \widetilde{\mathcal{L}}_E, \widetilde{\mathcal{H}}_E$ and $\widetilde{\mathcal{D}}_E$. Note that S itself need not be regular. We also obtain results concerning the extension of (one-sided) congruences, which we apply to (one-sided) congruences on maximal subgroups of regular semigroups.

If S has an inverse subsemigroup U of E-regular elements, such that $E \subseteq U$ and U intersects every $\widetilde{\mathcal{H}}_E$-class exactly once, then we say that U is an *inverse skeleton* of S. We give some natural examples of semigroups possessing inverse skeletons and examine a situation where we can build an inverse skeleton in a $\widetilde{\mathcal{D}}_E$-simple monoid. Using these techniques, we show

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that a reasonably wide class of $\widetilde{\mathcal{D}}_E$-simple monoids can be decomposed as Zappa-Szép products. Our approach can be immediately applied to obtain corresponding results for bisimple inverse monoids.

1 Introduction

Decomposing semigroups using Green's relations is the classical approach to semigroup structure. Regular $\mathcal{D}$-classes are particularly well understood, given that the left and right translations afforded by Green's lemmas result in Green's theorem, which states that the $\mathcal{H}$-class of an element a is a subgroup if and only if $a\,\mathcal{H}\,a^2$. For non-regular $\mathcal{D}$-classes, indeed for non-regular semigroups, an approach using Green's relations is not always the most appropriate. As an alternative, one can make use of the extensions $\mathcal{K}^*$ of Green's relations $\mathcal{K}$, where $K \in \{R, L, H, D\}$ or the yet wider relations $\widetilde{\mathcal{K}}_E$, where E is a set of idempotents. The aim of this current paper is to take an approach that is something of a synthesis: we study semigroups possessing E-regular elements, where an element a of a semigroup S is *E-regular* if a has an inverse a° such that $aa^\circ, a^\circ a$ lie in $E \subseteq E(S)$.

After recalling the definitions of $\widetilde{\mathcal{K}}_E$ in Section 2, we show that where E-regular elements exist in particular places, then analogues of Green's lemmas hold where $\mathcal{K}$ is replaced by $\widetilde{\mathcal{K}}_E$. With some extra conditions on our semigroup we also have an analogue of Green's theorem. Namely, we show that under these conditions, if $a\,\widetilde{\mathcal{H}}_E\,a^2$, then $\widetilde{H}_E^a$, the $\widetilde{\mathcal{H}}_E$-class of a, is a monoid with identity from E. In Section 3 we show that if $\widetilde{\mathcal{H}}_E$ is a congruence on a semigroup S, then any right congruence on the submonoid $\widetilde{H}_E^e$, where $e \in E$, can be extended to a congruence on S. We also have a result for two sided congruences, with some further restrictions on S. We stress that for regular semigroups with $E = E(S)$ we have $\widetilde{\mathcal{K}}_E = \mathcal{K}^* = \mathcal{K}$, so our results can be immediately applied to maximal subgroups of regular semigroups.

In Section 4 we introduce the idea of an *inverse skeleton* U of a semigroup S. Here U is an inverse subsemigroup of E-regular elements, such that $E \subseteq U$ and U intersects every $\widetilde{\mathcal{H}}_E$-class exactly once (it follows that $E = E(U)$). We examine some conditions under which we obtain skeletons from monoids having a particular submonoid L of the $\widetilde{\mathcal{L}}_E$-class of the identity. A monoid with such a submonoid L is called *special*. Our most complete results are for restriction monoids, which for convenience we briefly define in Section 2.

Finally, in Section 5, we investigate the decomposition of some special $\widetilde{\mathcal{D}}_E$-simple monoids as what we refer to as *Zappa-Szép products*, also known as *general products*. The concept of Zappa-Szép product was first studied for groups by Neumann [15] and subsequently by Zappa [19] and Casadio [1]. The Zappa-Szép product of two groups is a natural generalisation of the notion of semidirect product, which itself extends that of direct product. Szép initiated the study of Zappa-Szép products in settings other than groups in [17, 18]. Zappa-Szép products for monoids have been further investigated by, for example, Kunze [10–12] and Lavers [13]. In particular, Kunze gave applications of Zappa-Szép products to translational hulls, Bruck-Reilly extensions and Rees matrix semigroups. In this paper we focus on a result of Kunze [10] for the Bruck-Reilly extension $\mathrm{BR}(M,\theta)$ of a monoid M, showing that $\mathrm{BR}(M,\theta)$ is a Zappa-Szép product of $\mathbb{N}^0$ under addition and a semidirect product $M \rtimes \mathbb{N}^0$. Certainly $\mathrm{BR}(M,\theta)$ is special, with L isomorphic to $\mathbb{N}^0$. We put Kunze's result in more general framework and prove in particular that a special $\widetilde{\mathcal{D}}_E$-simple restriction monoid can be decomposed in an analogous way. Again, our results apply immediately to inverse monoids.

A few words on notation. Given a semigroup S, we denote by $E(S)$ its set of idempotents and by E a subset of $E(S)$. We assume that the reader is familiar with Green's relations and their associated preorders and the starred versions thereof. Details of the latter and of the relations $\widetilde{\mathcal{K}}_E$, which we define below, can be found in the notes [6].

2 The relations $\widetilde{\mathcal{R}}_E, \widetilde{\mathcal{L}}_E$ and analogues of Green's lemmas

We recall that the relation $\leq_{\widetilde{\mathcal{R}}_E}$ on S is defined by the rule that for all $a, b \in S$ we have $a \leq_{\widetilde{\mathcal{R}}_E} b$ if and only if

$$\{e \in E : eb = b\} \subseteq \{e \in E : ea = a\}.$$

It is clear that $\leq_{\widetilde{\mathcal{R}}_E}$ is a pre-order on S, that is, a relation that is reflexive and transitive. The associated equivalence relation is denoted by $\widetilde{\mathcal{R}}_E$. Thus for any $a, b \in S$ we have $a\,\widetilde{\mathcal{R}}_E\,b$ if and only if a and b have same set of left identities in E. It is easy to see that $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$. The relations $\leq_{\widetilde{\mathcal{L}}_E}$ and $\widetilde{\mathcal{L}}_E$ are defined dually so that clearly $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_E$. Note that any $e \in E$ is a left (right) identity for its $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class). If S is regular and $E = E(S)$, then the foregoing inclusions are replaced by equalities. More generally, if $e, f \in E$ then $e\,\widetilde{\mathcal{R}}_E\,f$ if and only if $e\,\mathcal{R}\,f$ and $e\,\widetilde{\mathcal{L}}_E\,f$ if and only if $e\,\mathcal{L}\,f$. In general, however, the inclusions are strict.

We will show that, under certain circumstances, $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ behave like $\mathcal{R}$ and $\mathcal{L}$. In general, however, they do not. The first thing to observe is that, unlike $\mathcal{R}$ and $\mathcal{R}^*$, the relation $\widetilde{\mathcal{R}}_E$ need not be a left congruence; of course the dual remark is also true. We say that S *satisfies the Congruence Condition* (C) *with respect to* E (or, more simply, S *satisfies* (C)) if $\widetilde{\mathcal{R}}_E$ is a left congruence and $\widetilde{\mathcal{L}}_E$ is a right congruence. A second observation is that, as is the case with $\mathcal{R}^*$ and $\mathcal{L}^*$, the relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ need not commute. We denote by $\widetilde{\mathcal{H}}_E$ and $\widetilde{\mathcal{D}}_E$ the intersection and join of $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ respectively. Note that from the previous remark, it is not usually the case that $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. Deviating slightly from standard terminology, we will denote the $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class, $\widetilde{\mathcal{H}}_E$-class, $\widetilde{\mathcal{D}}_E$-class) of any $a \in S$ by $\widetilde{R}^a_E$ ($\widetilde{L}^a_E$, $\widetilde{H}^a_E$, $\widetilde{D}^a_E$).

One class of semigroups having the congruence condition is the class of restriction semigroups. Left restriction semigroups form a variety of unary semigroups, that is, semigroups equipped with an additional unary operation, denoted by $^+$. The identities that define a left restriction semigroup S are:

$$a^+a = a,\ a^+b^+ = b^+a^+,\ (a^+b)^+ = a^+b^+ \text{ and } ab^+ = (ab)^+a.$$

Putting $E = \{a^+ : a \in S\}$, it is easy to see that E is a semilattice, the *semilattice of projections* of S. Dually, right restriction semigroups form a variety of unary semigroups, where in this case the unary operation is denoted by *. A bi-unary semigroup S (that is, a semigroup with two unary operations) which is both left restriction and right restriction and which also satisfies the linking identities

$$(a^+)^* = a^+ \text{ and } (a^*)^+ = a^*$$

is called a *restriction semigroup.* We remark that an inverse semigroup is restriction, where we define $a^+ = aa^{-1}$ and $a^* = a^{-1}a$. If a restriction semigroup S has an identity element 1, then it is easy to see that $1^+ = 1^* = 1$. Such a restriction semigroup is naturally called a *restriction monoid.*

A restriction semigroup satisfies (C) (with respect to E) and is such that the $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class) of an element a contains a unique element of E, namely a^+ (a^*). Restriction semigroups and their one sided versions have been studied from various point of view and under different names since the 1960s. They were formerly called *weakly E-ample semigroups*, to emphasize that the class naturally extends the class of *ample semigroups.* For detailed studies of the basic properties of these structures and a historical overview, the reader is referred to [5] and [6].

The next remark is folklore, but worth stating as a lemma.

Lemma 2.1. *If S satisfies (C), then $\widetilde{H}_E^e$ is a monoid with identity e, for any $e \in E$.*

Lemma 2.2. *Let S be a semigroup satisfying (C). Then if $a, b \in S$ and $a \,\widetilde{\mathcal{R}}_E\, e \,\widetilde{\mathcal{L}}_E\, b$, for some $e \in E$, we have that $a \,\widetilde{\mathcal{L}}_E\, ba \,\widetilde{\mathcal{R}}_E\, b$.*

Proof. As $a \,\widetilde{\mathcal{R}}_E\, e$ and $\widetilde{\mathcal{R}}_E$ is left congruence, we have $ba \,\widetilde{\mathcal{R}}_E\, be = b$. Dually, $ba \,\widetilde{\mathcal{L}}_E\, a$.

$\boxed{e}$		a
b		ba

□

Definition 2.3. An element $c \in S$ is *E-regular* if c has an inverse c° such that $cc^\circ, c^\circ c \in E$.

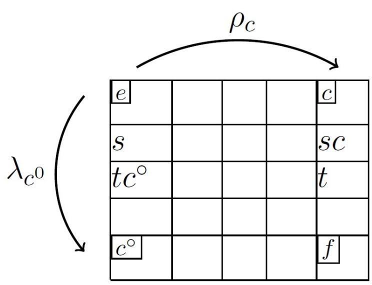
We emphasise that the notation c° will always be used with this meaning. Of course, if c is E-regular, then so is c°. Observe that if $c \in S$ is E-regular and $g, h \in E$ with $g \, \widetilde{\mathcal{R}}_E \, c \, \widetilde{\mathcal{L}}_E \, h$, then $cc^\circ \, \mathcal{R} \, c \, \widetilde{\mathcal{R}}_E \, g$ and $c^\circ c \, \mathcal{L} \, c \, \widetilde{\mathcal{L}}_E \, h$, so that by an earlier remark, $cc^\circ \, \mathcal{R} \, g$ and $c^\circ c \, \mathcal{L} \, h$. It follows from standard results for regular elements that c has an inverse c' such that $cc' = g$ and $c'c = h$. It is also easy to see (in view of earlier remarks concerning idempotents), that if $h, k \in S$ are E-regular, then $h \, \widetilde{\mathcal{K}}_E \, k$ if and only if $h \, \mathcal{K} \, k$, where K is R, L or H.

We first show that analogues of Green's Lemmas hold with $\mathcal{R}$, $\mathcal{L}$ replaced by $\widetilde{\mathcal{R}}_E$, $\widetilde{\mathcal{L}}_E$ where there is a suitable E-regular element.

Lemma 2.4. *Suppose that $\widetilde{\mathcal{L}}_E$ is a right congruence and S has an E-regular element c such that $e = cc^\circ$ and $f = c^\circ c$. Then the right translations*

$$\rho_c : \widetilde{L}^e_E \to \widetilde{L}^f_E \quad \textit{and} \quad \rho_{c^\circ} : \widetilde{L}^f_E \to \widetilde{L}^e_E$$

are mutually inverse $\widetilde{\mathcal{R}}_E$-class preserving bijections.



Proof. Notice that $e \, \mathcal{R} \, c \, \mathcal{L} \, f$. Let $s \in \widetilde{L}^e_E$. Since $\widetilde{\mathcal{L}}_E$ is a right congruence, $sc \, \widetilde{\mathcal{L}}_E \, ec = c$ so there is a map $\rho_c : \widetilde{L}^e_E \to \widetilde{L}^f_E$ defined by $s\rho_c = sc$. Now $s = se = scc^\circ \, \mathcal{R} \, sc$, so that certainly ρ_c is $\widetilde{\mathcal{R}}_E$-class preserving. Dually, $\rho_{c^\circ} : \widetilde{L}^f_E \to L^e_E$ is $\widetilde{\mathcal{R}}_E$-class preserving.

For any $s \in \widetilde{L}^e_E$ and $t \in \widetilde{L}^f_E$ we have $s = se = s(cc^\circ) = s\rho_c\rho_{c^\circ}$ and similarly, $t = t\rho_{c^\circ}\rho_c$, so that ρ_c and ρ_{c° are mutually inverse on the specified domains. □

Note that we are not assuming that the $\widetilde{\mathcal{D}}_E$-class depicted above is an "egg-box", since as $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ need not commute, some of the cells may be empty.

For convenience we now state the dual of Lemma 2.4.

Lemma 2.5. *Suppose that $\widetilde{\mathcal{R}}_E$ is a left congruence and S has an E-regular element c such that $e = cc^\circ$ and $f = c^\circ c$. Then the left translations*

$$\lambda_{c^\circ} : \widetilde{R}^e_E \to \widetilde{R}^f_E \quad \textit{and} \quad \lambda_c : \widetilde{R}^f_E \to \widetilde{R}^e_E$$

are mutually inverse $\widetilde{\mathcal{L}}_E$-class preserving bijections.

Corollary 2.6. *Let S be a semigroup with (C). Let c be an E-regular element of S such that $e = cc^\circ$ and $f = c^\circ c$. Then $\widetilde{H}^e_E \cong \widetilde{H}^f_E$.*

Proof. By Lemmas 2.4 and 2.5, $\rho_c : \widetilde{H}^e_E \to \widetilde{H}^c_E$ and $\lambda_{c^\circ} : \widetilde{H}^c_E \to \widetilde{H}^f_E$ are bijections. Now For any $x, y \in \widetilde{H}^e_E$ we have

$$\begin{aligned} (xy)\rho_c\lambda_{c^\circ} &= c^\circ xyc \\ &= c^\circ xcc^\circ yc \qquad \text{as } cc^\circ = e \\ &= (x\rho_c\lambda_{c^\circ})(y\rho_c\lambda_{c^\circ}). \end{aligned}$$

Thus $\rho_c\lambda_{c^\circ}$ is an isomorphism and hence $\widetilde{H}^e_E \cong \widetilde{H}^f_E$. □

If we have enough E-regular elements, then we can say much more than in Corollary 2.6. First, we recall that S is *weakly E-abundant* if every $\widetilde{\mathcal{R}}_E$- and every $\widetilde{\mathcal{L}}_E$-class of S contains an idempotent of E. Clearly a regular semigroup S is weakly $E(S)$-abundant; on the other hand, any monoid is weakly $\{1\}$-abundant. A less extreme example is $M_n(R)$, the monoid of $n \times n$ matrices over a principal ideal domain, under matrix multiplication [4]. In such a monoid we have $\widetilde{\mathcal{R}}_E = \mathcal{R}^*$ and $\widetilde{\mathcal{L}}_E = \mathcal{L}^*$, where $E = E\big(M_n(R)\big)$, and further, every $\mathcal{H}^*$-class contains a regular element. The reader will see other natural examples as the article progresses.

Lemma 2.7. *If every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element, then S is weakly E-abundant. Moreover if S has (C), then $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ (so that $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E$) and if $a, b \in S$ with $a \,\widetilde{\mathcal{D}}_E\, b$, then $|\widetilde{H}_E^a| = |\widetilde{H}_E^b|$.*

Proof. The first statement is clear. Suppose that $a, c \in S$ with $a \,\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E\, c$.

				$b^\circ b$
a		bb°		b
$cb^\circ a$				c

There exists an E-regular $b \in S$ such that $a \,\widetilde{\mathcal{R}}_E\, b \,\widetilde{\mathcal{L}}_E\, c$. Choose an inverse b° of b such that $bb^\circ, b^\circ b \in E$. Notice that $c \,\widetilde{\mathcal{L}}_E\, b^\circ b$ and $a \,\widetilde{\mathcal{R}}_E\, bb^\circ$. Using (C), $cb^\circ a \,\widetilde{\mathcal{R}}_E\, cb^\circ b = c$ and $cb^\circ a \,\widetilde{\mathcal{L}}_E\, bb^\circ a = a$. Then $a \,\widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E\, c$. Together with the dual argument we have that $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. In view of the remarks following Definition 2.3, the proof of the final statement follows easily from Lemmas 2.4 and 2.5. □

Green's theorem, a pivot of classical semigroup theory, states that if $k \in S$ and $k \,\mathcal{H}\, k^2$, then H_k is a group. We now consider semigroups with (C) such that the analogue of Green's theorem holds, by which we mean, if $k \,\widetilde{\mathcal{H}}_E\, k^2$, then $\widetilde{H}_E^k$ is a monoid with identity an element of E: in view of Lemma 2.1, this is equivalent to containing an element of E.

The set of idempotents $E(T)$ of any semigroup T may be endowed with the two pre-orders $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$, under which it has the structure of a *biordered set*; if T is regular, then $E(T)$ is a *regular biordered set.*

Conversely, any biordered set is the biordered set of idempotents of a semigroup, which is regular if E is regular [3, 14]. Suppose now that S is our semigroup with $E \subseteq E(S)$; [14, Theorem 1.3] gives necessary and sufficient conditions such that E generates a regular subsemigroup $S' = \langle E \rangle$ of S such that $E(S') = E$. Clearly, if these conditions hold, and if $h \in S'$ with $h \, \widetilde{\mathcal{H}}_E \, h^2$ in S, then as $E \subseteq S'$ we have $h \, \widetilde{\mathcal{H}}_E \, h^2$ in S'. It follows that $h \, \mathcal{H} \, h^2$ in S' so that $h \, \mathcal{H} \, u$ in S' for some $u \in E(S') = E$. Certainly then $\widetilde{H}_E^h$ (in either S or S') contains u.

To obtain a more general result, we need to introduce the following concept.

Definition 2.8. We say that $E \subseteq E(S)$ is *closed under E-conjugation* if for any $e \in E$ and E-regular $c \in S$ (with $cc^\circ, c^\circ c \in E$), if $cec^\circ \in E(S)$, then $cec^\circ \in E$.

Notice that the above definition is symmetric, since $(c^\circ)^\circ = c$.

Lemma 2.9. *Let S be a restriction semigroup, let $c \in S$ be E-regular and let $e \in E$. Then cec° (and hence also $c^\circ ec$) lie in E.*

Proof. Let c, e be as above. Then

$$cec^\circ = (ce)^+ cc^\circ \in E$$

as E is a semilattice. □

The next lemma follows the pattern for regular semigroups, as stated in [7, Result 2]. However, we need a little care as E need not consist of all idempotents of S.

Lemma 2.10. *The E-regular elements of S form a subsemigroup T with $E = E(T)$ if and only if ef is E-regular for any $e, f \in E$, and E is closed under E-conjugation.*

Proof. Let T denote the set of E-regular elements of S. The direct statement is clear.

Conversely, suppose that ef is E-regular for any $e, f \in E$, and E is closed under E-conjugation. Let $h, k \in T$ and choose inverses h°, k° of h and k respectively, such that $hh^\circ, f = h^\circ h, e = kk^\circ, k^\circ k \in E$. Let u be an inverse of fe such that $ufe, feu \in E$. It is easy to check that $k^\circ uh^\circ$ is an inverse of hk. We then have $(hk)(k^\circ uh^\circ) \in E(S)$ and

$$(hk)(k^\circ uh^\circ) = hf(kk^\circ)uh^\circ = h(feu)h^\circ,$$

so that $(hk)(k^\circ uh^\circ) \in E$ as $feu \in E$ and E is closed under E-conjugation. Similarly, $(k^\circ uh^\circ)hk \in E$. Thus $hk \in T$ as required. □

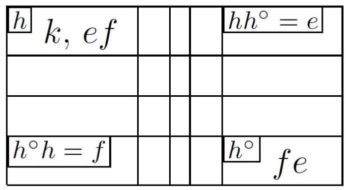
Corollary 2.11. *Suppose that ef is E-regular for any $e, f \in E$, and E is closed under E-conjugation. If $h \in S$ is E-regular and $h \,\widetilde{\mathcal{H}}_E\, h^2$, then $\widetilde{H}_E^h$ contains an idempotent of E; hence if S satisfies (C), then $\widetilde{H}_E^h$ is a monoid with identity from E.*

Proof. From Lemma 2.10 we have that the E-regular elements of S form a subsemigroup T with $E = E(T)$. Certainly $h, h^2 \in T$ with $h \,\widetilde{\mathcal{H}}_E\, h^2$ in T. Then $h \,\mathcal{H}\, h^2$ in T so that as $E = E(T)$ we have $\widetilde{H}_E^h$ (in either T or S) contains an idempotent of E. □

Whereas the previous result uses Green's theorem, the next does not, but has rather restrictive hypotheses.

Lemma 2.12. *Suppose that $E \subseteq E(S)$ is a band, every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element, $\widetilde{\mathcal{H}}_E$ is a congruence and S satisfies (C). Then for $k \in S$ with $k \,\widetilde{\mathcal{H}}_E\, k^2$, we have $E \cap \widetilde{H}_E^k \neq \emptyset$.*

Proof. Notice that as $\widetilde{\mathcal{H}}_E$ is a congruence and $k\,\widetilde{\mathcal{H}}_E\,k^2$, we have that $\widetilde{H}_E^k$ is a subsemigroup.



By hypothesis there exists an E-regular element $h \in \widetilde{H}_E^k$ such that $hh^\circ = e, h^\circ h = f \in E$. Then

$$h^\circ = h^\circ h h^\circ \,\widetilde{\mathcal{H}}_E\, h^\circ h h h^\circ = fe \in E.$$

By Lemma 2.2, $ef \in \widetilde{H}_E^k$ and $ef \in E$ as E is a band. Hence $E \cap \widetilde{H}_E^k \neq \emptyset$. □

3 Extending congruences

Let M be a subsemigroup of a semigroup S and let ρ be a congruence (respectively, right congruence) on M. We denote by $\widetilde{\rho}$ (respectively, $\bar{\rho}$) the congruence (respectively, right congruence) on S generated by ρ. We briefly review the circumstances under which $\rho = \widetilde{\rho} \cap (M \times M)$ or $\rho = \bar{\rho} \cap (M \times M)$, where $M = \widetilde{H}_E^e$ for some $e \in E$, in the context of the conditions discussed in this article.

Definition 3.1. A subsemigroup M of a semigroup S has the *(right) congruence extension property* in S if for any (right) congruence ρ on M we have

$$\rho = \widetilde{\rho} \cap (M \times M) \text{ (respectively, } \rho = \bar{\rho} \cap (M \times M)).$$

Lemma 3.2. *Let S be a weakly E-abundant semigroup with (C). Suppose that $\widetilde{\mathcal{H}}_E$ is a congruence. Let $e \in E$. Then $M = \widetilde{H}_E^e$ has the right congruence extension property in S.*

Proof. Let ρ be a right congruence on M. Clearly $\rho \subseteq \bar{\rho} \cap (M \times M)$. Let $a \in M, b \in S$ and suppose $a \,\bar{\rho}\, b$. Then either $a = b$ (so that clearly $a \,\rho\, b$) or there exists a sequence

$$a = c_1 t_1,\ d_1 t_1 = c_2 t_2,\ \cdots, d_n t_n = b$$

for some $n \in \mathbb{N}$, where $(c_i, d_i) \in \rho$, $t_i \in S$, $1 \leq i \leq n$ (see, for example, [9, Chapter 1]). As $a, c_1, d_1, \cdots, c_n, d_n \in M$, which has identity e, we have

$$a = c_1 t_1',\ d_1 t_1' = c_2 t_2',\ \cdots, d_n t_n' = b \quad \text{where } t_i' = e t_i.$$

Since $\widetilde{\mathcal{H}}_E$ is a congruence we have

$$a = c_1 t_1' \,\widetilde{\mathcal{H}}_E\, e t_1' = t_1' \,\widetilde{\mathcal{H}}_E\, d_1 t_1' = c_2 t_2' \,\widetilde{\mathcal{H}}_E\, e t_2' = t_2' \,\widetilde{\mathcal{H}}_E\, \cdots \,\widetilde{\mathcal{H}}_E\, e t_n' = t_n'.$$

We conclude that $t_1', \cdots, t_n' \in M$ and so $b \in M$ and $a \,\rho\, b$. Hence M has the right congruence extension property. □

Note that what we have shown above is something a little stronger than claimed, namely that $\bar{\rho}$ saturates M.

Corollary 3.3. *Let S be a regular semigroup such that $\mathcal{H}$ is a congruence. Then for any $e \in E(S)$, the maximal subgroup H_e has the right congruence extension property.*

Let M be a subsemigroup of S and let ρ be a congruence on M. We say that ρ is *closed under E-conjugation* if for $u, v \in M$ with $u \,\rho\, v$ and for any E-regular $c \in S$ with $cuc^\circ, cvc^\circ \in M$, we have $cuc^\circ \,\rho\, cvc^\circ$; if $E = E(S)$, we simply say that ρ is *closed under conjugation.*

Proposition 3.4. *Let S be a semigroup with (C) such that every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element, $\widetilde{\mathcal{H}}_E$ is a congruence and if $k \,\widetilde{\mathcal{H}}_E\, k^2$, then $\widetilde{H}_E^k$ contains an idempotent of E. Let $e \in E$ and $M = \widetilde{H}_E^e$ and let ρ be a congruence on M. Then*

$$\rho = \widetilde{\rho} \cap (M \times M),$$

if and only if ρ is closed under E-conjugation.

Proof. It is clear that if $\rho = \widetilde{\rho} \cap (M \times M)$, then ρ is closed under E-conjugation.

Conversely, suppose that ρ is closed under E-conjugation. Let $a \in M, b \in S$ and suppose that

$$a = cpd,\; cqd = b,$$

where $(p, q) \in \rho$ and $c, d \in S^1$. As $p \,\widetilde{\mathcal{H}}_E^{e}\, q$ and $\widetilde{\mathcal{H}}_E$ is a congruence, we see that $b \in M$. It follows that

$$a = c'pd',\; c'qd' = b,$$

where $c' = ece$ and $d' = ede$. Then

$$a \leq_{\widetilde{\mathcal{R}}_E} c' \leq_{\widetilde{\mathcal{R}}_E} e \,\widetilde{\mathcal{R}}_E\, a,$$

so that $a \,\widetilde{\mathcal{R}}_E\, c'$. Dually, $a \,\widetilde{\mathcal{L}}_E\, d'$.

e a	v°			c' u
u°				f
d' v u^*	g			$d'c'$ w

From the comments following Definition 2.3, there exist E-regular elements $u \in \widetilde{H}_E^{c'}$ and $v \in \widetilde{H}_E^{d'}$ such that $uu^\circ = e, u^\circ u = f \in E$ and $v^\circ v = e, vv^\circ = g \in E$. Now $vu \in \widetilde{R}_E^v \cap \widetilde{L}_E^u$ by Lemma 2.2 and $vu \,\widetilde{\mathcal{H}}_E\, d'c'$. Since

$$uv \,\widetilde{\mathcal{H}}_E\, c'd' = c'ed' \,\widetilde{\mathcal{H}}_E\, c'pd' = a \,\widetilde{\mathcal{H}}_E\, e$$

we have

$$vuvu \,\widetilde{\mathcal{H}}_E\, veu \,\widetilde{\mathcal{H}}_E\, vu.$$

By assumption, there exists an idempotent $w \in E \cap \widetilde{H}_E^{d'c'}$. Let $u^* \in \widetilde{H}_E^{d'}$ be an inverse of u such that $uu^* = e$ and $u^*u = w$. Then

$$a = c'wpwd' = (c'u^*)(upu^*)(ud') \text{ and } b = c'wqwd' = (c'u^*)(uqu^*)(ud').$$

Now $u^* \,\widetilde{\mathcal{H}}_E\, d'$ gives that $c'u^* \,\widetilde{\mathcal{H}}_E\, c'd' \,\widetilde{\mathcal{H}}_E\, e$, so $c'u^* \in M$ and similarly $u \,\widetilde{\mathcal{H}}_E\, c'$ gives that $ud' \,\widetilde{\mathcal{H}}_E\, c'd' \,\widetilde{\mathcal{H}}_E\, e$, so that $ud' \in M$. Further,

$$upu^* = e(upu^*)e \,\widetilde{\mathcal{H}}_E\, (c'u^*)(upu^*)(ud') = a \in M,$$

and similarly, $uqu^* \in M$. Since ρ is closed under E-conjugation it follows that $upu^* \,\rho\, uqu^*$ and so $a \,\rho\, b$.

Now consider $h \in M, k \in S$ with $h \,\widetilde{\rho}\, k$. Either $h = k$ (so that certainly

$h\,\rho\,k$), or h is connected to k via a ρ-sequence

$$h = c_1p_1d_1, c_1q_1d_1 = c_2p_2d_2, \cdots, c_nq_nd_n = k,$$

for some $n \in \mathbb{N}$, where $(p_i, q_i) \in \rho$, $c_i, d_i \in S^1$, $1 \leq i \leq n$ (see, for example, [8, Chapter 1]). It follows from the above that $c_iq_id_i \in M$ and $h\,\rho\,c_iq_id_i$ for $1 \leq i \leq n$. Hence $h\,\rho\,k$ and

$$\rho = \widetilde{\rho} \cap (M \times M).$$

□

Corollary 3.5. *Let S be a regular semigroup such that $\mathcal{H}$ is a congruence. Let $G = H_e$ be the maximal subgroup with identity $e \in E(S)$. Then for any right congruence ρ on G we have $\rho = \widetilde{\rho} \cap (G \times G)$ if and only if ρ is closed under conjugation.*

Note that if E is a band, then from Lemma 2.12, the remaining hypotheses of Proposition 3.4 will guarantee that $\widetilde{H}^k_E$ contains an idempotent of E.

In the following, M is a monoid with identity e.

Example 3.6. Let B be a band. With $E = \{e\} \times B$, the direct product $M \times B$ satisfies the hypotheses of Proposition 3.4.

The next three examples are essentially folklore, but they can all be found in [2].

Example 3.7. Let $S = \mathcal{B}^\circ(M, I)$ be a 'Brandt' semigroup. That is,

$$S = (I \times M \times I) \cup \{0\}$$

with multiplication given by

$$(i,m,j)(j,n,k) = (i,mn,k),$$

all other products being 0. Then with

$$E = \{(i,1,i) : i \in I\} \cup \{0\}$$

we have that for any $(i,m,j),(k,n,l) \in M$

$$(i,m,j)\,\widetilde{\mathcal{R}}_E\,(k,n,l) \text{ if and only if } i = k$$

and

$$(i,m,j)\,\widetilde{\mathcal{L}}_E\,(k,n,l) \text{ if and only if } j = l.$$

It follows that S is restriction with distinguished semilattice E, $\widetilde{\mathcal{H}}_E$ is a congruence on S and with

$$U = \{(i,e,j) : i,j \in I\} \cup \{0\}$$

we have that U is an inverse subsemigroup of E-regular elements, intersecting every $\widetilde{\mathcal{H}}_E$-class exactly once. In particular, S satisfies the hypotheses of Proposition 3.4.

Example 3.8. Let $S = \mathrm{BR}(M,\theta)$, where $\theta : M \to M$ is a monoid morphism. That is,

$$S = \mathbb{N}^0 \times M \times \mathbb{N}^0$$

and multiplication is given by

$$(m,a,n)(h,b,k) = (m-n+u, a\theta^{u-n}b\theta^{u-h}, k-h+u) \text{ where } u = \max\,(n,h).$$

With

$$E = \{(m,e,m) : m \in \mathbb{N}^0\}$$

we have that for any $(m,a,n),(h,b,k) \in S$,

$$(m,a,n)\,\widetilde{\mathcal{R}}_E\,(h,b,k) \text{ if and only if } m = h$$

and

$$(m,a,n)\,\widetilde{\mathcal{L}}_E\,(h,b,k) \text{ if and only if } n = k.$$

It is then easy to see that $\widetilde{\mathcal{H}}_E$ is a congruence on S and S is restriction. Moreover, with

$$U = \{(m,e,n) : m,n \in \mathbb{N}^0\}$$

we have that U is an inverse subsemigroup of E-regular elements of S intersecting every $\widetilde{\mathcal{H}}_E$-class exactly once. In particular, S satisfies the hypotheses of Proposition 3.4. Note that S is a monoid with identity $(0,e,0)$.

Note that the assumption in [2] that the image of θ is contained in $\widetilde{H}_E^1$, is not needed for the above.

Example 3.9. Let $S = \mathrm{BR}(M,\mathbb{Z},\theta)$ be the extended Bruck-Reilly extension of a monoid M. The underlying set is

$$S = \mathbb{Z} \times M \times \mathbb{Z}$$

and the semigroup operation on S is defined as in Example 3.8. The semigroup S has the same properties as in that example, with the exception of being a monoid.

Example 3.10. Let $S = [Y; S_\alpha; \chi_{\alpha,\beta}]$ be a strong semilattice Y of monoids $S_\alpha, \alpha \in Y$, with connecting morphims $\chi_{\alpha,\beta}$ for $\alpha \geqslant \beta$. Denoting the identity of S_α by e_α we have that S is restriction with

$$E = \{e_\alpha : \alpha \in Y\} \cong Y,$$

and the S_αs are the $\widetilde{\mathcal{H}}_E$-classes. Certainly then $\widetilde{\mathcal{H}}_E$ is a congruence on S and S satisfies the hypotheses of Proposition 3.4.

4 Semigroups with skeletons

We continue to examine semigroups with 'enough' E-regular elements, now moving towards decompositions of such semigroups. It is clear from Lemma 2.7 that if every $\widetilde{\mathcal{H}}_E$-class of a semigroup S with (C) contains an E-regular element, and $e\,\widetilde{\mathcal{D}}_E\,a$ where $e \in E$, then every element of $\widetilde{H}_E^a$ has a unique decomposition as upv, where u, v are fixed E-regular elements and $p \in \widetilde{H}_E^e$. For results leading further to structure theorems, we will concentrate in this section on the case where E is a semilattice.

Definition 4.1. Let $V \subseteq W$ be subsets of a semigroup S such that W is a union of $\widetilde{\mathcal{H}}_E$-classes. We say that V is an $\widetilde{\mathcal{H}}_E$-*transversal* of W if

$$|V \cap \widetilde{H}_E^a| = 1 \quad \text{for all } a \in W.$$

Lemma 4.2. *Let E be a semilattice and let $c \in S$ be E-regular. Then there is only one choice of c°. Moreover, if $d \in S$ is E-regular and $c\,\widetilde{\mathcal{H}}_E\,d$, then $c^\circ\,\widetilde{\mathcal{H}}_E\,d^\circ$.*

Proof. If c°, c' are both inverses of c with $cc^\circ, cc', c^\circ c, c'c \in E$, then we have

$$c \,\widetilde{\mathcal{L}}_E\, c^\circ c \,\widetilde{\mathcal{L}}_E\, c'c \text{ and } cc^\circ \,\widetilde{\mathcal{R}}_E\, c \,\widetilde{\mathcal{R}}_E\, cc'.$$

Since E is a semilattice, any $\widetilde{\mathcal{R}}_E$-class or $\widetilde{\mathcal{L}}_E$-class contains at most one idempotent of E, so that $c^\circ c = c'c = e$ and $cc^\circ = cc' = f$ say. Thus $c^\circ, c' \in R_e \cap L_f$ so that (as any $\mathcal{H}$-class contains at most one inverse of c) we have $c^\circ = c'$.

The proof of the second statement is similar. □

Clearly the above shows that if E is a semilattice and $c \in S$ is E-regular, then $(c^\circ)^\circ = c$. We recall that S is said to be *weakly E-adequate* if S is weakly E-abundant and E is a semilattice. In this case there is a unique idempotent in the $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class) of $a \in S$, which we denote by a^+ (a^*, respectively).

Note 4.3. Let S be a weakly E-adequate semigroup and let $c \in S$ be E-regular. Then

$$c \,\widetilde{\mathcal{R}}_E\, c^+ \,\widetilde{\mathcal{R}}_E\, cc^\circ,$$

so that we must have $c^+ = cc^\circ$ and similarly $c^* = c^\circ c$. Hence also $(c^\circ)^+ = c^\circ c$ and $(c^\circ)^* = cc^\circ$.

Proposition 4.4. *Let S be weakly E-adequate with $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$, and let $e \in E$. Suppose there is an $\widetilde{\mathcal{H}}_E$-transversal L of $\widetilde{L}^e_E$ such that every $c \in L$ is E-regular, and $e \in L$. Then:*

1. *$R = \{c^\circ : c \in L\}$ is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}^e_E$;*
2. *$D = LR$ is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{D}^e_E$;*
3. *if S has (C), then every element of $\widetilde{D}^e_E$ has a unique decompostion as cpd°, where $c, d \in L$ and $p \in \widetilde{H}^e_E$.*

Proof. (1) Let $c \in L$. As E is a semilattice and $c\,\widetilde{\mathcal{L}}_E\,e$, we must have that $e = c^\circ c$ so that $e\,\widetilde{\mathcal{R}}_E\,c^\circ$. From Lemma 4.2, clearly R intersects any $\widetilde{\mathcal{H}}_E$-class at most once. On the other hand, let $a \in \widetilde{R}^e_E$. Then $a\,\widetilde{\mathcal{L}}_E\,f \in E$ and as $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$, we have that $f\,\widetilde{\mathcal{R}}_E\,c$ for some $c \in L$. It follows that $a\,\widetilde{\mathcal{H}}_E\,c^\circ$, so that R is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}^e_E$.

(2) It is clear from Lemma 2.2 that for any $c, d \in L$ we have $cd^\circ \in \widetilde{R}^c_E \cap \widetilde{L}^{d^\circ}_E$. Since $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$, it follows that D is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{D}^e_E$, as required.

(3) This follows from Lemmas 2.4 and 2.5.

□

We anticipate that Proposition 4.4 can be used to develop structure theorems for classes of weakly E-adequate semigroups analogous to those for inverse semigroups.

Definition 4.5. Let U be an inverse subsemigroup of S consisting of E-regular elements such that $E \subseteq U$. If U is an $\widetilde{\mathcal{H}}_E$-transversal of S, then U is an *inverse skeleton* of S.

Example 4.6. The semigroups of Examples 3.7, 3.8 and 3.10 all have inverse skeletons, with E being the skeleton in Example 3.10.

Lemma 4.7. *Let S be a semigroup containing an inverse skeleton U. Then $E = E(U)$ is a semilattice, S is weakly E-adequate and if in addition S has (C), we have $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$.*

Proof. We are given that $E \subseteq E(U)$. If $u \in E(U)$, then as u is E-regular, $u\,\mathcal{R}\,uu^\circ \in E$. We are given that $E(U)$ is a semilattice and so

$u = uu^\circ \in E$. The remainder of the lemma is immediate from Lemma 2.7. $\square$

Naturally, we say that S is $\widetilde{\mathcal{D}}_E$*-simple* if it is a single $\widetilde{\mathcal{D}}_E$-class.

Theorem 4.8. *Let S be a $\widetilde{\mathcal{D}}_E$-simple weakly E-adequate monoid with $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. Suppose there is a submonoid $\widetilde{\mathcal{H}}_E$-transversal L of $\widetilde{L}^1_E$ such that every $c \in L$ is E-regular and for all $c \in L$, $e \in E$ we have $cec^\circ, c^\circ ec \in E$. Let*

$$R = \{c^\circ : c \in L\}.$$

1. *R is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}^1_E$;*
2. *$RL \subseteq \widetilde{R}^1_E \cup \widetilde{L}^1_E$ if and only if E is a chain;*
3. *if S is restriction then $U = \langle R \cup L \rangle$ is an inverse subsemigroup of S with $E(U) = E$;*
4. *if S is restriction and $RL \subseteq R \cup L$, then $U = LR$ and U is an inverse skeleton for S.*

Proof. From the condition that $cec^\circ, c^\circ ec \in E$ for all $c \in L$, and the fact that E is a semilattice, it is easy to see that for any $u, v \in R \cup L$ we have uv is E-regular with suitable inverse $v^\circ u^\circ$.

(1) From Proposition 4.4, we know that R is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}^1_E$. Let $c, d \in L$ so that $c^\circ, d^\circ \in R$. From the above, cd is E-regular with $(cd)^\circ = d^\circ c^\circ$. As $cd \in L$ we have $d^\circ c^\circ \in R$. Clearly, $1 = 1^\circ \in R$, so that R is a submonoid.

(2) Let $e, f \in E$ and let $c, d \in L$ be such that $cc^\circ = e, dd^\circ = f$. As above, $c^\circ d$ is E-regular with $(c^\circ d)^\circ = d^\circ c$. We have $c^\circ d \in \widetilde{R}^1_E$ if and only if $1 = c^\circ dd^\circ c$, which implies (multiplying on the front by

c and the back by c°) that $e = efe$ so that $e \leq f$. On the other hand, if $e \leq f$, then $c^\circ d \,\widetilde{\mathcal{R}}_E\, c^\circ ef = c^\circ e = c^\circ \,\widetilde{\mathcal{R}}_E\, 1$. Similarly, we see that $c^\circ d \in \widetilde{L}^1_E$ if and only if $f \leq e$. Statement (2) follows.

(3) Let $u = x_1 x_2 \ldots x_k \in U$, where $x_i \in L \cup R$ for $1 \leq i \leq n$. We show by induction on k that u is E-regular with $u^\circ = x_k^\circ \ldots x_1^\circ$. Clearly this is true for $k = 1$ and we commented above that this is true for $k = 2$.

Suppose now that $k \geqslant 3$ and the result is true for words in U of shorter length. Our inductive hypothesis gives that $x_1 \ldots x_{k-1}$ is E-regular with inverse $x_{k-1}^\circ \ldots x_1^\circ$. Then

$$\begin{aligned}
&(x_1 \cdots x_k)(x_k^\circ \cdots x_1^\circ)(x_1 \cdots x_k) \\
&\qquad = (x_1 \cdots x_{k-1})(x_k x_k^\circ)[(x_{k-1}^\circ \cdots x_1^\circ)(x_1 \cdots x_{k-1})]x_k \\
&\qquad = (x_1 \cdots x_{k-1})[(x_{k-1}^\circ \cdots x_1^\circ)(x_1 \cdots x_{k-1})](x_k x_k^\circ)x_k \\
&\qquad = x_1 \cdots x_{k-1} x_k
\end{aligned}$$

and

$$(x_1 \cdots x_k)(x_k^\circ \cdots x_1^\circ) = x_1(x_2 \cdots x_k x_k^\circ \cdots x_2^\circ)x_1^\circ \in E$$

by induction and hypothesis. Together with the dual argument, we obtain that $u = x_1 \cdots x_k$ is E-regular with $u^\circ = x_k^\circ \cdots x_1^\circ$.

Certainly $E \subseteq E(U)$. To show that U is inverse, we use the fact that S is restriction. Let $e \in E(U)$. Then

$$e^+ = ee^\circ = eee^\circ = ee^+$$

so that using the identity $xy^+ = (xy)^+x$ we have

$$e^+ = ee^+ = (ee)^+e = e^+e = e,$$

so that $E(U) = E$. Hence $E(U)$ is a semilattice and U is inverse.

(4) To see that $U = LR$, let $u \in U$. Since R and L are submonoids, we can write $u = l_1r_1l_2r_2 \cdots l_mr_m$ where $l_1, \ldots, l_m \in L$ and $r_1, \ldots, r_m \in R$ and m is least with respect to such a decomposition of u. If $m \geqslant 2$, then either $r_1l_2 \in R$ or $r_1l_2 \in L$, so that as

$$u = l_1(r_1l_2r_2) \cdots l_mr_m = (l_1r_1l_2)r_2 \cdots l_mr_m$$

we have violated the minimality of m. Hence $m = 1$ and $U = LR$. From Proposition 4.4, U is an $\widetilde{\mathcal{H}}_E$-transversal of S, so that U is an inverse skeleton of S.

□

Example 4.9. Let $S = \mathrm{BR}(M, \theta)$ and put

$$L = \{(m, e, 0) : m \in \mathbb{N}^0\}.$$

We have that L is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}^1_E$ consisting of E-regular elements and $S \times S = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. With

$$R = \{(0, e, m) : m \in \mathbb{N}^0\} = \{(m, e, 0)^\circ : m \in \mathbb{N}^0\}$$

we see that $RL \subseteq R \cup L$. Then U defined as in Theorem 4.8 coincides with U as given in Example 3.8.

5 $\widetilde{\mathcal{D}}_E$-simple monoids and Zappa-Szép products

We build on the results of previous sections to show how certain $\widetilde{\mathcal{D}}_E$-simple restriction monoids decompose as Zappa-Szép products of submonoids. In particular, we show how Kunze's [10] result for the Bruck-Reilly extension of a monoid may be put into a general framework.

For the convenience of the reader we begin by recalling the basic definitions relating to Zappa-Szép products.

Definition 5.1. Let U and V be monoids and suppose that we have maps

$$V \times U \to U,\ (t, s) \mapsto t \cdot s \text{ and } V \times U \to V,\ (t, s) \mapsto t^s$$

such that for all $s, s' \in U, t, t' \in V$:

(ZS1) $tt' \cdot s = t \cdot (t' \cdot s)$; (ZS5) $t \cdot 1_U = 1_U$;

(ZS2) $t \cdot (ss') = (t \cdot s)(t^s \cdot s')$; (ZS6) $t^{1_U} = t$;

(ZS3) $(t^s)^{s'} = t^{ss'}$; (ZS7) $1_V \cdot s = s$;

(ZS4) $(tt')^s = t^{t' \cdot s} t'^s$; (ZS8) $1_V^s = 1_V$.

Define a binary operation on $U \times V$ by

$$(s, t)(s', t') = (s(t \cdot s'), t^{s'} t').$$

Then $U \times V$ is a monoid, most recently referred to as the (external) *Zappa-Szép product of* U *and* V and denoted by $U \bowtie V$.

It is clear that $U \bowtie V$ contains submonoids $U' = U \times \{1_V\}$ and $V' = \{1_U\} \times V$ such that every element of $U \bowtie V$ has a unique expresssion as $u'v'$ where $u \in U', v \in V'$. Thus $U \bowtie V$ is the internal Zappa-Szép product of U' and V', where we say that a monoid S is the *internal Zappa-Szép product* of submonoids U and V if $S = UV$ and every element of S

has a *unique* expression as $uv, u \in U, v \in V$. In this case, writing

$$vu = (v \cdot u)(v^u)$$

we have that U and V act on each other satisfying (ZS1)–(ZS8) and $S \cong U \bowtie V$ under the isomorphism $uv \mapsto (u, v)$ [13].

Note that if one of the above actions is trivial (that is, by identity maps), then the second action is by morphisms, and we obtain the semidirect product $U \rtimes V$ (if U acts trivially) or $U \ltimes V$ (if V acts trivially).

Definition 5.2. Let S be a monoid. We say that S is *special* if there is a submonoid $\widetilde{\mathcal{H}}_E$-transversal L of $\widetilde{L}^1_E$ such that every $c \in L$ is E-regular.

Example 5.3. We have observed in Example 4.9 that $S = \mathrm{BR}(M, \theta)$ is special with

$$L = \{(m, e, 0) : n \in \mathbb{N}^0\}$$

being a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}^1_E$. Moreover, $\widetilde{\mathcal{H}}_E$ is a congruence on S.

Theorem 5.4. *Let S be a weakly E-adequate monoid with (C). Then S is $\widetilde{\mathcal{D}}_E$-simple with $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ and special if and only if S is the internal Zappa-Szép product of L and $\widetilde{R}^1_E$, where L is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}^1_E$.*

Proof. Suppose that S is the internal Zappa-Szép product of L and $\widetilde{R}^1_E$, where L is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}^1_E$.

Let $a, b \in S$ and write $a = lr$, $b = l'r'$ where $l, l' \in L$ and $r, r' \in \widetilde{R}^1_E$. Then $lr', l'r \in S$,

$$a = lr \,\widetilde{\mathcal{R}}_E\, lr' \,\widetilde{\mathcal{L}}_E\, l'r' = b$$

and

$$a = lr\,\widetilde{\mathcal{L}}_E\,l'r\,\widetilde{\mathcal{R}}_E\,l'r' = b.$$

Thus $\widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = S \times S$. Finally we need to show that L consists of E-regular elements. For this let $l \in L$ and write $l^+ = uv$ where $u \in L$ and $v \in \widetilde{R}^1_E$. Then $u\,\widetilde{\mathcal{R}}_E\,l$ so that $u = l$, since $|L \cap \widetilde{H}^a_E| = 1$ for all $a \in L$.

$\boxed{1}$		$v = l^\circ$	
$\boxed{l = u}$		$l^+ = uv$	

Therefore $l^+ = lv$ and $l = l\,1 = l^+l = l(vl)$ and $vl \in \widetilde{H}^1_E$ by Lemma 2.2. By uniqueness of factorisation, $vl = 1$. Thus $v = vlv$ and $lv, vl \in E$, so that l is E-regular as required. Thus S is special.

Conversely, suppose that $\widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = S \times S$ and S is special. Let $s \in S$. Then $1\,\widetilde{\mathcal{L}}_E\,l\,\widetilde{\mathcal{R}}_E s$ for some $l \in L$ and as l is E-regular we have $s = l^+s = ll^\circ s$. Now observe that $l^\circ s\,\widetilde{\mathcal{R}}_E\,l^\circ l = 1$ so that $l^\circ s \in \widetilde{R}^1_E$. To see that this factorisation is unique, suppose that $s = lr = kt$ where $l, k \in L$ and $r, t \in \widetilde{R}^1_E$. Now $\widetilde{\mathcal{R}}_E$ is a left congruence, so that $l\,\widetilde{\mathcal{R}}_E\,k$, giving $l = k$. As l is E-regular, we have $1 = l^\circ l$ and we deduce that $r = t$. Thus S is the internal Zappa-Szép product of L and $\widetilde{R}^1_E$. $\square$

We now examine the actions in the situation where the hypotheses of Theorem 5.4 hold. For $r \in \widetilde{R}^1_E$ and $l \in L$ we have

$$rl = (rl)^+rl = dd^\circ rl$$

where $d \in L$. Observe now that $d^\circ rl\,\widetilde{\mathcal{R}}_E\,d^\circ(rl)^+ = d^\circ dd^\circ = d^\circ\widetilde{\mathcal{R}}_E\,1$. It follows that

$$r \cdot l = d \text{ and } r^l = d^\circ rl \text{ where } rl \,\widetilde{\mathcal{R}}_E\, d \in L.$$

We explain these actions with the help of an egg-box picture.

1		r	$r^l = d^\circ rl$
l			
$r \cdot l = d$			rl

We can proceed further in Theorem 5.4 to decompose $\widetilde{R}^1_E$ as a Zappa-Szép product, under the additional hypothesis that for all $c \in L$ and $e \in E$ we have $cec^\circ, c^\circ ec \in E$. Recall from Theorem 4.8 that this guarantees that $R = \{c^\circ : c \in L\}$ is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}^1_E$.

Theorem 5.5. *Let S be a weakly E-adequate monoid with (C) such that S is $\widetilde{\mathcal{D}}_E$-simple with $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ and special. Suppose in addition that for all $c \in L$ and $e \in E$ we have $cec^\circ, c^\circ ec \in E$. Then $\widetilde{R}^1_E$ is the internal Zappa-Szép product of $\widetilde{H}^1_E$ and R.*

It follows that $\widetilde{R}^1_E \cong \widetilde{H}^1_E \bowtie R$. Further, if $\widetilde{\mathcal{H}}_E$ is a congruence on S, then the action of $\widetilde{H}^1_E$ on R is trivial and $\widetilde{R}^1_E \cong \widetilde{H}^1_E \rtimes R$.

Proof. Let $t \in \widetilde{R}^1_E$. For $r \in R$ with $r \,\widetilde{\mathcal{H}}_E\, t$, we have $rr^\circ = 1$ and $r^\circ r = f \in E$ and certainly $f \,\widetilde{\mathcal{L}}_E\, r$. From Lemma 2.4, $\rho_r : \widetilde{H}^1_E \to \widetilde{H}^r_E$ is a bijection. Thus every element of $\widetilde{R}^1_E$ has a unique decomposition as hr for some $h \in \widetilde{H}^1_E$ and $r \in R$, that is, $\widetilde{R}^1_E = \widetilde{H}^1_E R$ is the internal Zappa-Szép product of $\widetilde{H}^1_E$ and R.

It follows that $\widetilde{R}^1_E \cong \widetilde{H}^1_E \bowtie R$. We now examine the mutual actions of $\widetilde{H}^1_E$ and R. Let $h \in \widetilde{H}^1_E, r \in R$ and let $t \in R$ be such that $rh \,\widetilde{\mathcal{H}}_E\, t$, so that $rh \,\widetilde{\mathcal{L}}_E\, f = t^\circ t$. Then $rh = (rh)f = (rh)(t^\circ t)$ and $rht^\circ \in \widetilde{H}^1_E$, again by Lemma 2.4. Hence $r \cdot h = rht^\circ$ and $r^h = t$:

$\boxed{1}$ $h \quad r \cdot h = rht^\circ$		$\boxed{r}$	$\boxed{t = r^h}\, rh$	
$\boxed{t^\circ}$			$\boxed{t^\circ t}$	

Finally, if $\widetilde{\mathcal{H}}_E$ is congruence, then $rh\,\widetilde{\mathcal{H}}_E\,r1 = r$, so that $t = r$ and $r^h = r$. □

6 Some applications and examples

If S is such that every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element and S has (C), then we have noted in Lemma 2.7 that $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. Moreover, if S is special and restriction, then we immediately see from Lemma 2.9 that for all $c \in L$ and $e \in E$ we have $cec^\circ, c^\circ ec \in E$. In particular, if S is an inverse monoid, then certainly with $E = E(S)$, S is restriction, every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element and $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ (since $\widetilde{\mathcal{K}}_E = \mathcal{K}$, for all relevant K). We thus immediately deduce from Theorems 5.4 and 5.5 the following: notice that we have reverted to the more usual notation of K_a for the $\mathcal{K}$-class of $a \in S$.

Theorem 6.1. *Let S be an inverse monoid. Then S is bisimple and special if and only if S is the internal Zappa-Szép product of L and R_1, where L is a submonoid $\mathcal{H}$-transversal of L_1. Moreover, in this case, R_1 is the internal Zappa-Szép product of H_1 and R where $R = \{r^{-1} : r \in L\}$, and is a semidirect product if $\mathcal{H}$ is a congruence.*

Now we deduce [10, Section 5.4].

Corollary 6.2. *Let $S = BR(M, \theta)$. Then with*

$$L = \{(n, e, 0) : n \in \mathbb{N}^0\} \text{ and } R = \{(0, e, n) : n \in \mathbb{N}^0\}$$

we have that

$$S \cong \mathbb{N}^0 \bowtie (M \rtimes \mathbb{N}^0).$$

Proof. We have observed that S is restriction and special with L and R as given. Moreover, $S \times S = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{H}}_E$ is a congruence. From Theorems 5.4 and 5.5 we have $S \cong L \bowtie (\widetilde{H}^1_E \rtimes R)$ and then as $L \cong \mathbb{N}^0$, $\widetilde{H}^1_E \cong M$ and $L \cong \mathbb{N}^0$, we deduce the result.

We now consider the relevant actions. For $(n, e, 0) \in L$ and $(0, a, m) \in \widetilde{R}^1_E$, with $k = \max(m, n)$ we have

$$(0, a, m)(n, e, 0) = (k - m, a\theta^{k-m}, k - n)$$

so that from the recipe in Theorem 5.4 we have

$$(0, a, m) \cdot (n, e, 0) = (k - m, e, 0) \text{ and } (0, a, m)^{(n,e,0)} = (0, a\theta^{k-m}, k - n).$$

Considering now the action of R on $\widetilde{H}^1_E$ we have

$$(0, e, m) \cdot (0, a, 0) = (0, e, m)(0, a, 0)(m, e, 0) = (0, a\theta^m, 0).$$

Using the natural isomorphisms $(n, e, 0) \mapsto n, (0, e, m) \mapsto m$ and $(0, a, 0) \mapsto a$ we have that $\mathbb{N}^0$ acts on S by

$$m \cdot a = a\theta^m$$

giving the semidirect product $S \rtimes \mathbb{N}^0$ and then $S \rtimes \mathbb{N}^0$ and $\mathbb{N}^0$ act on eachother mutually by

$$(a, m) \cdot n = k - m \text{ and } (a, m)^n = (a\theta^{k-m}, k - n).$$

□

Of course, the above can be applied to the bicyclic monoid (with M trivial) or to bisimple inverse ω-semigroups (with M a group).

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