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On (po-)torsion free and principally weakly (po-)flat S-posets

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Abstract. In this paper, we first consider (po-)torsion free and principally weakly (po-)flat S-posets, specifically we discuss when (po-)torsion freeness implies principal weak (po-)flatness. Furthermore, we give a counterexample to show that Theorem 3.22 of Shi is incorrect. Thereby we present a correct version of this theorem. Finally, we characterize pomonoids over which all cyclic S-posets are weakly po-flat.

1 Introduction and Preliminaries

A monoid S is called a *pomonoid* if it is also a poset whose partial order is compatible with the binary operation. A right S*-poset* A_S is a right S-act A_S equipped with a partial order $\leq$ and, in addition, $s \leq t$ implies $as \leq at$, and $a \leq b$ implies $as \leq bs$ for all $s, t \in S$ and $a, b \in A$. A sub S-poset of a right S-poset A is a subset of A that is closed under the S-action. Ideals are the same as for acts. Moreover, S-poset morphisms or simply S-poset maps are monotone maps between S-posets which preserve actions. The

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class of S-posets and S-poset maps form a category, denoted by $S - POS$, which comprises the main background of this work. For an account on this category and categorical notions used in this paper, the reader is referred to [2].

Let A_S be a right S-poset. An S-poset congruence θ on A_S is a right S-act congruence with the property that the S-act A/θ can be made into an S-poset in such a way that the natural map $A \longrightarrow A/\theta$ is an S-poset map. For an S-act congruence θ on A_S we write $a \leq_\theta a'$ if the so-called θ-chain

$$a \leq a_1\theta b_1 \leq a_2\theta b_2 \leq \ \ldots \ \leq a_n\theta b_n \leq a'$$

from a to a' exists in A, where $a_i, b_i \in A,\ 1 \leq i \leq n$. It can be shown that an S-act congruence θ on a right S-poset A is an S-poset congruence if and only if $a\theta a'$ whenever $a \leq_\theta a' \leq_\theta a$.

Let A_S be an S-poset and $H \subseteq A \times A$. Define a relation $\alpha(H)$ on A by $a \leq_{\alpha(H)} b$ if and only if $a \leq b$ or there exist $n \geq 1, (c_i, d_i) \in H, s_i \in S, 1 \leq i \leq n$ such that

$$a \leq c_1 s_1 \quad d_1 s_1 \leq c_2 s_2 \quad \ldots \quad d_n s_n \leq b.$$

The relation $\nu(H)$ given by $a\ \nu(H)\ b$ if and only if $a \leq_{\alpha(H)} b \leq_{\alpha(H)} a$ is called *the S-poset congruence induced by H*. The order relation on $A/\nu(H)$ is given by $[a]_{\nu(H)} \leq [b]_{\nu(H)}$ if and only if $a \leq_{\alpha(H)} b$. Moreover, $\theta(H) = \nu(H \cup H^{-1})$ for any $H \subseteq A \times A$ is called the *S-poset congruence generated by H*. In particular, the S-poset congruence generated by (a, b) is called a *monocyclic* S-poset congruence and is denoted by $\theta(a, b)$. Also, the S-poset congruence induced by (a, b) is called an *induced monocyclic* S-poset congruence and is denoted by $\nu(a, b)$.

Let A be a right S-poset and B a left S-poset. The order relation on $A_S \otimes {}_S B$ can be described as follows: $a \otimes b \leq a' \otimes b'$ holds in $A_S \otimes {}_S B$ if and only if there exist $s_1, \ldots, s_n, t_1, \ldots, t_n \in S, a_1, \ldots, a_n \in A_S$, $b_2, \ldots, b_n \in {}_S B$ such that

$$\begin{array}{rclcrcl} a & \leq & a_1 s_1 & & & & \\ a_1 t_1 & \leq & a_2 s_2 & & s_1 b & \leq & t_1 b_2 \\ & \vdots & & & & \vdots & \\ a_n t_n & \leq & a' & & s_n b_n & \leq & t_n b'. \end{array}$$

Specifically, when $B = Sb$ and $b = b'$, we can replace all b_i by b in the above scheme. Moreover, $a \otimes b = a' \otimes b'$ if and only if $a \otimes b \leq a' \otimes b'$ and

$a' \otimes b' \leq a \otimes b$. A right S-poset A_S is *weakly po-flat* if $as \leq a't$ implies that $a \otimes s \leq a' \otimes t$ in $A_S \otimes {}_S(Ss \cup St)$ for $a, a' \in A_S, s, t \in S$. A right S-poset A_S is *principally weakly po-flat* if $as \leq a's$ implies $a \otimes s \leq a' \otimes s$ in $A_S \otimes {}_S Ss$ for $a, a' \in A_S$ and $s \in S$. Weak flatness and principal weak flatness can be defined similarly, replacing $\leq$ by equality.

An element c of a pomonoid S is called *right po-cancellable* if, for all $s, t \in S$, $sc \leq tc$ implies $s \leq t$. A right S-poset A_S is called *po-torsion* (*torsion*) *free* if $ac \leq a'c$ ($ac = a'c$) implies $a \leq a'$ ($a = a'$) for $a, a' \in A_S$ and a right po-cancellable (cancellable) element c of S.

Also, recall that if A_S is a sub S-poset of B_S, the amalgamated coproduct of two copies of B_S over a proper sub S-poset A_S, denoted by $B_S \sqcup^A B_S$, is defined by $B_S \sqcup^A B_S = (\{x, y\} \times (B \setminus A)) \cup (\{z\} \times A)$, where $x, y, z \notin B$, with the right S-action given by

$$(w, u)s = \begin{cases} (w, us), & us \notin A,\ w \in \{x, y\} \\ (z, us), & us \in A \end{cases}$$

and the order defined as:

$$\begin{aligned}(w_1, u_1) \leq (w_2, u_2) \Longleftrightarrow & \ (w_1 = w_2 \text{ and } u_1 \leq u_2) \text{ or} \\ & (w_1 \neq w_2,\ u_1 \leq u \leq u_2 \text{ for some } u \in A).\end{aligned}$$

In particular, for a proper right ideal I of a pomonoid S, we denote the amalgamated coproduct of two copies of S over I by $A(I)$.

The flatness properties of S-posets, specially (po-)torsion freeness and principal weak (po-)flatness, have been studied in [1], [3], [11], [12]. In [1] and [10], (po-)torsion free and principally weakly (po-)flat Rees factor S-posets were discussed. In [8] the authors characterized pomonoids over which all S-posets are (po-)torsion free or principally weakly (po-)flat. In [9], the flatness properties of $A(I)$ were investigated.

In this paper, we will continue the study of (po-)torsion freeness and principal weak (po-)flatness of S-posets. First our attention is restricted to (po-)torsion free monocyclic S-posets of the form $S/\theta(wt, t)$. Then in Section 2 we present some conditions under which (po-)torsion freeness implies principal weak (po-)flatness. In Section 3, we give a counterexample for Theorem 3.22 of [12], and further we present a correct form of this theorem. Finally, we characterize the pomonoids over which all cyclic S-posets are weakly po-flat.

A subpomonoid K of a pomonoid S is called *convex* if $K = [K]$ where $[K] = \{x \in S \mid \exists\ p, q \in K,\ p \leq x \leq q\}$. If X is a subset of a poset P, $(X] := \{p \in P \mid \exists x \in X, p \leq x\}$ is the *down-set of* X; $[X) := \{p \in P \mid \exists x \in X, x \leq p\}$ is the *up-set of* X. First we recall the (po-)torsion freeness of Rees factor S-posets.

Proposition 1.1 ([1]). *Let K_S be a convex, proper right ideal of a pomonoid S. Then S/K_S is torsion free if, and only if, for every $s \in S$ and every right cancellable $c \in S$, $sc \in K$ implies $s \in K$.*

In the following, we present another version of Proposition 6 of [1] using the symbol $(K]$.

Proposition 1.2. *Suppose K_S is a proper, convex right ideal of a pomonoid S. Then S/K_S is po-torsion free if, and only if, for every $s \in S$ and every right po-cancellable element $c \in S$, $sc \in (K]$ implies $s \in (K]$ and $sc \in [K)$ implies $s \in [K)$.*

Similar to the case of acts, an element t of a pomonoid S is called *w-regular* for $w \in S$ if $wt \neq t$ and for any right cancellable element $c \in S$ and any $u \in S$, $uc \in tS$ implies $u\theta(wt, t)wu$. The following lemma is easily proved.

Lemma 1.3. *Let t, w be elements of a pomonoid S, $wt \neq t$ and $\rho = \theta(wt, t)$. If S/ρ is torsion free, then t is w-regular.*

Definition 1.4. Let t, w be elements of a pomonoid S, $wt \neq t$ and $\rho = \theta(wt, t)$. We call the element t *ordered w-regular* if for any right po-cancellable element c and any $u \in S$, $uc \in (tS]$ implies $u \leq_\rho wu$, and $uc \in [tS)$ implies $wu \leq_\rho u$.

Lemma 1.5. *Let t, w be elements of a pomonoid S, $wt \neq t$ and $\rho = \theta(wt, t)$. If t is ordered w-regular, then S/ρ is po-torsion free.*

Proof. Let c be a right po-cancellable element of S and $xc \leq_\rho yc$ for $x, y \in S$. By Proposition 1.4 of [3], there exist $n, m > 0$ such that $w^n xc \leq w^m yc$, $w^i xc \in (tS]$, and $w^j yc \in [tS)$ for each $0 \leq i < n$ and $0 \leq j < m$. Clearly, $w^n x \leq w^m y$. Moreover, since t is ordered w-regular, $w^i xc \in (tS]$ implies

$w^i x \leq_\rho w^{i+1} x$ for $0 \leq i < n$, and $w^j yc \in [tS)$ implies $w^{j+1} y \leq_\rho w^j y$ for $0 \leq j < m$. Therefore,

$$x \leq_\rho wx \leq_\rho w^2 x \leq_\rho \ldots \leq_\rho w^n x \leq_\rho w^m y \leq_\rho w^{m-1} y \leq_\rho \ldots \leq_\rho y,$$

and the proof is complete. □

2 When (po-)torsion free S-posets are principally weakly (po-)flat

In this section, we first characterize pomonoids S over which all (po-)torsion free right S-posets are principally weakly (po-)flat. Then we give some conditions under which all (po-)torsion free right Rees factor S-posets are principally weakly (po-)flat.

First see the following lemma.

Lemma 2.1 ([9])**.** *Let I be a right ideal of a pomonoid S. Then $A(I)$ is torsion free if and only if for each right cancellable element $c \in S$ and $s \in S$, $sc \in I$ implies that $s \in I$.*

In view of the previous lemma we have the following.

Corollary 2.2. *Let $I = \{s \in S|\ s$ is not right cancellable$\}$. Then $A(I)$ is torsion free.*

Corollary 2.3. *If $A(I)$ is torsion free for a proper right ideal I of S, then $I \subseteq \{s \in S|\ s$ is not right cancellable$\}$.*

In [6], Laan introduced the notion of a left almost regular monoid, and proved that all torsion free right S-acts are principally weakly flat if and only if S is left almost regular. Now, we shall prove an analogue of this result for S-posets.

Definition 2.4. An element s of a pomonoid S is called *left almost regular* if there exist right cancellable elements c_i and elements $s_i, r_i, r \in S$, $1 \leq i \leq n$, such that

$$\begin{aligned} s_1 c_1 &= s r_1 \\ s_2 c_2 &= s_1 r_2 \\ &\vdots \\ s_n c_n &= s_{n-1} r_n \\ s &= s_n r s. \end{aligned}$$

A pomonoid S is *left almost regular* if all its elements are left almost regular.

Theorem 2.5. *For any pomonoid S, the following statements are equivalent*:

(i) all torsion free right S-posets are principally weakly flat;

(ii) all finitely generated torsion free right S-posets are principally weakly flat;

(iii) S is left almost regular.

Proof. (i)$\Longrightarrow$ (ii): It is obvious.

(ii)$\Longrightarrow$ (iii): Let $s \in S$. Take I to be the set of all $t \in S$ with

$$\begin{array}{rcl} s_1c_1 & = & sr_1 \\ s_2c_2 & = & s_1r_2 \\ & \vdots & \\ tc_n & = & s_{n-1}r_n. \end{array} \tag{1}$$

Since $s \in I$, $I \neq \emptyset$. If $t \in I$ and $r \in S$, then we could add the row $(tr)\cdot 1 = t\cdot r$ to the above scheme, which follows that $tr \in I$, and so I is a right ideal of S. Next let us show that $A(I)$ is torsion free. Using Lemma 2.1, it suffices to show that if $s'c \in I$ for $s' \in S$ and a right cancellable element $c \in S$, then $s' \in I$. Let $s'c = t \in I$. Then the scheme (1) holds. Now, by adding the row $s'c = t \cdot 1$ to the scheme (1), we get $s' \in I$, so $A(I)$ is torsion free. By (ii), $A(I)$ is principally weakly flat. Then for $s \in I$, by Lemma 2.2 of [8], there exists $j \in I$ such that $s = js$. Therefore,

$$\begin{array}{rcl} s_1c_1 & = & sr_1 \\ s_2c_2 & = & s_1r_2 \\ & \vdots & \\ jc_m & = & s_{m-1}r_m \\ s & = & j\cdot 1\cdot s, \end{array}$$

and so S is left almost regular, as required.

(iii)$\Longrightarrow$ (i): The proof is similar to that of Theorem 4.6.5 of [4]. □

Regularly right almost regular pomonoids were introduced in [13] where the authors proved that these are precisely pomonoids over which all regularly divisible right S-posets are regularly principally weakly injective. The definition of a regularly left almost regular pomonoid is a dual form of the definition of a regularly right almost regular pomonoid, as follows.

Definition 2.6. An element s of a pomonoid S is called *regularly left almost regular* if there exist $s_i, r_i, s'_i, r \in S$ and right po-cancellable elements c_i, $1 \leq i \leq n$, such that

$$\begin{array}{rclcrcl} s_1c_1 & \leq & sr_1 & \quad & sr_1 & \leq & s'_1c_1 \\ s_2c_2 & \leq & s_1r_2 & & s'_1r_2 & \leq & s'_2c_2 \\ & \vdots & & & & \vdots & \\ s_nc_n & \leq & s_{n-1}r_n & & s'_{n-1}r_n & \leq & s'_nc_n \\ s & = & s_ns & & s'_ns & = & s. \end{array}$$

A pomonoid S is *regularly left almost regular* if all its elements are regularly left almost regular.

Here, we consider when all po-torsion free right S-posets are principally weakly po-flat. Notice that the notation $D(S)$ is applied for S-poset $S \times S$.

Theorem 2.7. *The following are equivalent for a pomonoid S:*

(i) all po-torsion free right S-posets are principally weakly po-flat;

(ii) all finitely generated po-torsion free right S-posets are principally weakly po-flat;

(iii) S is regularly left almost regular.

Proof. (i)$\Longrightarrow$ (ii): It is clear.

(ii)$\Longrightarrow$ (iii): Let $s \in S$, and I be the set of all $(s_n, s'_n) \in D(S)$ with

$$\begin{array}{rclcrcl} s_1c_1 & \leq & sr_1 & \quad & sr_1 & \leq & s'_1c_1 \\ s_2c_2 & \leq & s_1r_2 & & s'_1r_2 & \leq & s'_2c_2 \\ & \vdots & & & & \vdots & \\ s_nc_n & \leq & s_{n-1}r_n & & s'_{n-1}r_n & \leq & s'_nc_n. \end{array}$$

Since $(s, s) \in I$, $I \neq \emptyset$. If $(s_n, s'_n) \in I$ and $r \in S$, then the above scheme together with $(s_nr)\cdot 1 \leq s_n \cdot r$ and $s'_n \cdot r \leq (s'_nr)\cdot 1$ implies $(s_n, s'_n)r \in I$. Thus I is a right sub S-poset of $D(S)$. First, we show that $B_S = D(S) \sqcup^I D(S)$ is po-torsion free. Suppose that $(w_1, (s', t'))c \leq (w_2, (s'', t''))c$ for a right po-cancellable element $c \in S$. If $w_1 = w_2$, from the fact that $D(S)$ is po-torsion free we deduce the result. Now, assume that $w_1 \neq w_2$. By the order relation of $A(I)$, there exists $(s_n, s'_n) \in I$ such that $(s', t')c \leq (s_n, s'_n) \leq (s'', t'')c$. So we have

$$\begin{array}{rclcrcl} s_1c_1 & \leq & sr_1 & & sr_1 & \leq & s'_1c_1 \\ s_2c_2 & \leq & s_1r_2 & & s'_1r_2 & \leq & s'_2c_2 \\ & \vdots & & & & \vdots & \\ s_nc_n & \leq & s_{n-1}r_n & & s'_{n-1}r_n & \leq & s'_nc_n \\ s'c & \leq & s_n & & s'_n & \leq & t''c, \end{array}$$

which means that $(s', t'') \in I$. Hence the inequalities $(s', t') \leq (s', t'') \leq (s'', t'')$ imply that $(w_1, (s', t')) \leq (w_2, (s'', t''))$, and so B_S is po-torsion free. By (ii), B_S is principally weakly po-flat. Then from $(s, s) \in I$ we see that $(x, (1, 1))s \leq (y, (1, 1))s$. Therefore, $(x, (1, 1)) \otimes s \leq (y, (1, 1)) \otimes s$ in $B_S \otimes {}_SSs$. This implies

$$\begin{array}{rclcrcl} (x, (1,1)) & \leq & (w_1, (u_1, v_1))k_1 & & & & \\ (w_1, (u_1, v_1))l_1 & \leq & (w_2, (u_2, v_2))k_2 & & k_1s & \leq & l_1s \\ & \vdots & & & & \vdots & \\ (w_n, (u_n, v_n))l_n & \leq & (y, (1,1)) & & k_ns & \leq & l_ns. \end{array}$$

One can easily prove that there exists $j = (s_n, s'_n) \in I$ such that $(1, 1)s \leq js \leq (1, 1)s$. Thus $s = s_ns$, $s'_ns = s$, and the result follows.

(iii)$\Longrightarrow$ (i): Let A_S be a po-torsion free S-poset, and $as \leq a's$ for $a, a' \in A$, $s \in S$. By (iii), there exist $s_i, r_i, s'_i, r \in S$ and right po-cancellable elements c_i, $1 \leq i \leq n$, such that

$$\begin{array}{rclcrcl} s_1c_1 & \leq & sr_1 & & sr_1 & \leq & s'_1c_1 \\ s_2c_2 & \leq & s_1r_2 & & s'_1r_2 & \leq & s'_2c_2 \\ & \vdots & & & & \vdots & \\ s_nc_n & \leq & s_{n-1}r_n & & s'_{n-1}r_n & \leq & s'_nc_n \\ s & = & s_ns & & s'_ns & = & s. \end{array}$$

So we have $as_1c_1 \leq asr_1 \leq a'sr_1 \leq a's'_1c_1$, and since A_S is po-torsion free, $as_1 \leq a's'_1$. Again, from $as_2c_2 \leq as_1r_2 \leq a's'_1r_2 \leq a's'_2c_2$ we get that $as_2 \leq a's'_2$. Continuing this process, we conclude that $as_n \leq a's'_n$. Therefore,

$$a \otimes s = a \otimes s_ns = as_n \otimes s \leq a's'_n \otimes s = a' \otimes s_ns = a' \otimes s$$

in $A_S \otimes {}_SSs$, and consequently A_S is principally weakly po-flat. $\square$

In the case of S-acts, torsion free right Rees factor S-acts are principally weakly flat if and only if S is left almost regular. But this is not the case for S-posets. Now, we define the weaker version of regularly left almost regular pomonoids to obtain the minor results on torsion free Rees factor S-posets.

Definition 2.8. An element s of a pomonoid S is called *ordered left (po-)almost regular* if there exist right (po-)cancellable elements c_i, c'_j and elements s_i, r_i, s'_j, r'_j, r, $r' \in S$, $1 \leq i \leq n$, $1 \leq j \leq m$, such that

$$\begin{array}{rcl} s_1c_1 & \leq & sr_1 \\ s_2c_2 & \leq & s_1r_2 \\ & \vdots & \\ s_nc_n & \leq & s_{n-1}r_n \\ s & \leq & s_nrs \end{array} \qquad \begin{array}{rcl} sr'_1 & \leq & s'_1c'_1 \\ s'_1r'_2 & \leq & s'_2c'_2 \\ & \vdots & \\ s'_{m-1}r'_m & \leq & s'_mc'_m \\ s'_mr's & \leq & s. \end{array}$$

A pomonoid S is *ordered left (po-)almost regular* if all its elements are ordered left (po-)almost regular.

Proposition 2.9. *If all torsion free Rees factor S-posets are principally weakly flat, then S is ordered left almost regular.*

Proof. Let $s \in S$, and K be the set of all $t \in S$ with

$$\begin{array}{rcl} s_1c_1 & \leq & sr_1 \\ s_2c_2 & \leq & s_1r_2 \\ & \vdots & \\ tc_n & \leq & s_{n-1}r_n \end{array} \qquad \begin{array}{rcl} sr'_1 & \leq & s'_1c'_1 \\ s'_1r'_2 & \leq & s'_2c'_2 \\ & \vdots & \\ s'_{m-1}r'_m & \leq & tc'_m, \end{array} \tag{2}$$

where c_i, c'_j are right cancellable elements of S. Since $s \in K$, $K \neq \emptyset$. It can be easily checked that K is a convex right ideal of S. Next we show that S/K is torsion free. Suppose that $s'c \in K$ for $s' \in S$ and a right cancellable element $c \in S$. Put $s'c = t$ for some $t \in K$. Then we have the scheme (2) holds. Now, from the scheme (2) and $s'c \leq t \cdot 1$, $t \cdot 1 \leq s'c$, it follows that $s' \in K$. By Proposition 1.1, S/K is torsion free. By the assumption, S/K is principally weakly flat. Then by Proposition 9 of [1], for $s \in K$, there exist $s', s'' \in K$ such that $s's \leq s \leq s''s$. Therefore, from the above scheme for s', s'', we can see that s is ordered left almost regular. □

Proposition 2.10. *If S is ordered left po-almost regular, then all po-torsion free Rees factor S-posets are principally weakly po-flat.*

Proof. Let S/K be po-torsion free where K is a proper convex right ideal of S, and $s \in S$, $k \in K$ such that $s \leq k$. We shall show that there exists $l \in K$ such that $s \leq ls$. Actually, s is ordered left po-almost regular, and so

$$\begin{array}{rclcrcl} s_1c_1 & \leq & sr_1 & \qquad & sr_1' & \leq & s_1'c_1' \\ s_2c_2 & \leq & s_1r_2 & & s_1'r_2' & \leq & s_2'c_2' \\ & \vdots & & & & \vdots & \\ s_nc_n & \leq & s_{n-1}r_n & & s_{m-1}'r_m' & \leq & s_m'c_m' \\ s & \leq & s_nrs & & s_m'r's & \leq & s, \end{array}$$

where c_i, c_j' are right po-cancellable elements of S. By Proposition 6 of [1], from $s_1c_1 \leq sr_1 \leq kr_1$ we have $s_1 \leq k'$ for some $k' \in K$. Then $s_2c_2 \leq s_1r_2 \leq k'r_2$ implies that $s_2 \leq k''$ for some $k'' \in K$. Continuing in this manner, we eventually reach $s_mr \leq l \in K$, and hence $s \leq ls$. Similarly, if $k \leq s$, one can show that there exists $l' \in K$ such that $l's \leq s$. Therefore, by Proposition 10 of [1] the result follows. □

By a similar argument of Proposition 2.9, we get the following corollary.

Corollary 2.11. *Let S be a pomonoid such that $[sS] = (sS] = [sS)$ for every $s \in S$. Then S is ordered left po-almost regular if and only if all po-torsion free Rees factor S-posets are principally weakly po-flat.*

3 Principally weakly po-flat S-posets

Shi presented in [12, Theorem 3.22] that over a right po-cancellable pomonoid, principal weak po-flatness, weak po-flatness and condition (P_w) are coincident. But in this proof it is only shown that weak po-flatness coincides with condition (P_w). Using this theorem, Liang and Luo in [7, Corollary 2.3] give more equivalent descriptions over a right po-cancellable pomonoid. In this section, we shall provide an example that deny these results, and further give the correct versions of them.

Recall that a pomonoid S is called *left PP*, if the sub S-poset Sx of the left S-poset ${}_SS$ is projective for every $x \in S$. A pomonoid S is called *left*

PSF if all principal left ideals of S are strongly flat as a left S-poset (that is, satisfies both conditions (P) and (E)). Clearly, every right po-cancellable pomonoid is left PP, and every left PP pomonoid is left PSF.

Recall also that an S-poset A_S satisfies *condition* (P_w) if, for all $a, b \in A$ and $s, t \in S$, $as \leq bt$ implies $a \leq a'u$, $a'v \leq b$ for some $a' \in A$, $u, v \in S$ with $us \leq vt$. An S-poset A_S is said to satisfy *condition* $(PWP)_w$ if, for all $a, b \in A_S$ and $s \in S$, $as \leq bs$ implies $a \leq a'u$ and $a'v \leq b$ for some $a' \in A_S$, $u, v \in S$ with $us \leq vs$.

Theorem 3.1 ([12, Theorem 3.22]). *Let S be a right po-cancellable pomonoid and A a right S-poset. Then the following statements are equivalent*:

(i) A satisfies condition (P_w);

(ii) A is principally weakly po-flat;

(iii) A is weakly po-flat.

Corollary 3.2 ([7, Corollary 2.3]). *Let S be a right po-cancellable pomonoid and A be a right S-poset. Then the following statements are equivalent*:

(i) A satisfies condition (P_w);

(ii) A satisfies condition $(WP)_w$;

(iii) A satisfies condition $(PWP)_w$;

(iv) A is principally weakly po-flat;

(v) A is weakly po-flat;

(vi) A is po-torsion free.

To give the example, we first recall the following result from [5].

Lemma 3.3 ([5]). *The diagonal S-poset $D(S)$ is weakly po-flat if and only if it is principally weakly po-flat and $Ss \cap (St] = \emptyset$ or for each $(as, a't)$ and $(bs, b't)$ in $H(s,t)$ there exists $(p,q) \in H(s,t)$ such that $(as, a't), (bs, b't) \in \widehat{S(p,q)}$, where $\widehat{S(p,q)} = \{(u,v) \in D(S) \mid \exists w \in S, u \leq wp, wq \leq v\}$ and $H(s,t) = \{(as, a't) \mid as \leq a't\}$ if $Ss \cap (St] \neq \emptyset$.*

Now we give a counterexample of Theorem 3.1 and Corollary 3.2.

Example 3.4. Let S denote the free monoid $\{a, b\}^*$. Define the relation $\leq$ on S by

$$w \leq z \text{ if, and only if, } w = z \text{ or } l(w) > l(z),$$

where $l(w)$ is the usual length of a word w. It can be checked that $\leq$ is a compatible order relation on S. First we show that S is a right po-cancellable pomonoid. Let $xc \leq yc$ for $x, y, c \in S$. If $xc = yc$, clearly $x = y$. Otherwise, $xc < yc$, that is $l(xc) > l(yc)$, which implies that $l(x) + l(c) > l(y) + l(c)$. Then $l(x) > l(y)$ which means $x < y$.

Next we show that $D(S)$ is principally weakly po-flat, but not weakly po-flat. Since S is a right po-cancellable pomonoid, S is a left PSF pomonoid. From Proposition 2.3 of [5] it follows that S^n is principally weakly po-flat for each $n \in \mathbf{N}$. In particular, $D(S)$ is principally weakly po-flat. However, to prove that $D(S)$ is not weakly po-flat we show that $D(S)$ does not satisfy the condition of Proposition 3.3. Clearly, $Sa \cap (Sb] \neq \emptyset$ and $(a^2, b), (ba, b) \in H(a, b)$. On the contrary, let there exist $(p, q) \in H(a, b)$ such that $(a^2, b), (ba, b) \in \widehat{S(p, q)}$. Then $p = ua,\ q = vb$ and $p \leq q$ for some $u, v \in S$. By the order relation on S, we have $l(p) > l(q)$ and $u \neq 1$. Moreover, $ba \leq wp,\ wq \leq b$ and $a^2 \leq zp,\ zq \leq b$ for some $w, z \in S$. If $l(ba) > l(wp)$, then $l(ba) = 2$ implies $l(wp) = l(wua) = 1$ and so $u = 1$, a contradiction. If $ba = wp = wua$, then $w = 1,\ u = b$, that is, $p = ba$. On the other hand, from $a^2 \leq zp,\ zq \leq b$, by a similar argument, we get $p = a^2$, which is a contradiction to $p = ba$. Therefore, $D(S)$ is not weakly po-flat.

Next we give correct versions of Theorem 3.1 and Corollary 3.2.

Proposition 3.5. *Let S be a right po-cancellable pomonoid and A_S be an S-poset. Then the following statements are equivalent*:

(i) A_S satisfies condition (P_w);

(ii) A_S satisfies condition $(WP)_w$;

(iii) A_S is weakly po-flat.

Proof. (i)$\Longrightarrow$ (ii) and (ii)$\Longrightarrow$ (iii) are obvious.

(iii)$\Longrightarrow$ (i): Let A_S be weakly po-flat. If $as \leq bt$ for $a, b \in A,\ s, t \in S$, then by Corollary 3.17 of [12], there exist $a' \in A,\ x, y \in E(S)$ (where $E(S)$ is the set of all idempotent elements of S) and $u, v \in S$ such that

$$xs = s,\ yt = t,\ us \leq vt,\ ax \leq a'u,\ a'v \leq by.$$

Since S is right po-cancellable, we have $x = y = 1$. So $a \leq a'u,\ a'v \leq b$ and $us \leq vt$. Therefore, A_S satisfies condition (P_w). □

Proposition 3.6. *Let S be a right po-cancellable pomonoid and A_S an S-poset. Then the following statements are equivalent*:

(ii) A_S satisfies condition $(PWP)_w$;
(ii) A_S is principally weakly po-flat;
(iii) A_S is po-torsion free.

Proof. (i)$\Longrightarrow$ (ii) is clear.

(ii)$\Longleftrightarrow$ (iii) follows by Theorem 3.21 of [12].

(ii)$\Longrightarrow$ (i): Let A_S be principally weakly po-flat. Assume $a, a' \in A$ and $s \in S$ are such that $as \leq a's$. Since S is also a left PP pomonoid, from Corollary 3.15 of [12], it follows that $es = s$ and $ae \leq a'e$ for some $e \in S$. Since S is right po-cancellable, we have $e = 1$. Therefore, $a \leq a \cdot 1$ and $a \cdot 1 \leq a'$. This means A_S satisfies condition $(PWP)_w$. □

At last, we turn our attention to the case that all cyclic S-posets are weakly po-flat.

Proposition 3.7. *The following are equivalent for a pomonoid S:*

(i) all cyclic S-posets are weakly po-flat;

(ii) all induced monocyclic S-posets are weakly po-flat;

(iii) all induced monocyclic S-posets are principally weakly po-flat, and S satisfies condition (R): for each $s, t \in S$ there exist $u, v \in S$ such that $s \leq_{\nu(s,t)} us \leq vt \leq_{\nu(s,t)} t$.

Proof. (i)$\Longrightarrow$ (ii) is clear.

(ii)$\Longrightarrow$ (iii): Suppose that $s, t \in S$ and set $\sigma = \nu(s,t)$. Then S/σ is weakly po-flat. Now, $s \leq_\sigma t$, and by Proposition 3.22 of [11], there exist $u, v \in S$ such that $us \leq vt$, $1 \leq_{ker\rho_s \vee \sigma} u$ and $v \leq_{ker\rho_t \vee \sigma} 1$. If $1 \leq_{ker\rho_s \vee \sigma} u$, then there exist $k_1, \ldots, k_n, l_1, \ldots, l_n \in S$ such that

$$\begin{array}{ccccccc} s \leq k_1 s & & l_1 s \leq k_2 s & & \cdots & & l_n s \leq us. \\ & k_1 \leq_\sigma l_1 & & k_2 \leq_\sigma l_2 & \cdots & k_n \leq_\sigma l_n & \end{array}$$

Therefore, $s \leq k_1 s \leq_\sigma l_1 s \leq k_2 s \leq_\sigma \ldots \leq_\sigma l_n s \leq us$. Indeed, $s \leq_\sigma us$. In a similar way, from $v \leq_{ker\rho_t \vee \sigma} 1$, we deduce that $vt \leq_\sigma t$, and so $s \leq_\sigma us \leq vt \leq_\sigma t$.

(iii)$\Longrightarrow$ (i): Let ρ be a right congruence on S, and $s \leq_\rho t$. Take $\sigma = \nu(s,t)$. By the assumption, there exist $u, v \in S$ such that $s \leq_\sigma us \leq vt \leq_\sigma t$. Since S/σ is principally weakly po-flat, by Proposition 3.20 of [11], $1 \leq_{ker\rho_s \vee \sigma} u$ and $v \leq_{ker\rho_t \vee \sigma} 1$. Now, $\sigma \leq \rho$ implies that $1 \leq_{ker\rho_s \vee \rho} u$ and $v \leq_{ker\rho_t \vee \rho} 1$. Thus by Proposition 3.22 of [11], S/ρ is weakly po-flat. □

Corollary 3.8. *If $S/\theta(s, s^2)$ is principally weakly po-flat, then s is an ordered regular element of S (that is, $ss's \leq s \leq ss''s$ for some $s', s'' \in S$).*

Proof. Let $\rho = \theta(s, s^2)$ and suppose S/ρ is principally weakly po-flat. Then we have $[s]_\rho = [s^2]_\rho$. If $[s]_\rho \leq [s^2]_\rho$, then we have $[1]_\rho \otimes s \leq [1]_\rho \otimes s^2$ in $S/\rho \otimes Ss$. By Lemma 3.5 of [11], either $s \leq s^2$, or there exist $a_i \in Ss$, $s_i, t_i \in \{s, s^2\}$, $i = 1, 2, \ldots, n$, such that

$$s \leq s_1 a_1, \quad t_1 a_1 \leq s_2 a_2, \quad \ldots, \quad t_n a_n \leq s^2.$$

Since $a_1 \in Ss$, we have $s \leq s_1 a_1 = ss''s$ for some $s'' \in S$. If $[s^2]_\rho \leq [s]_\rho$, then we have $[1]_\rho \otimes s^2 \leq [1]_\rho \otimes s$ in $S/\rho \otimes Ss$, it follows that $ss's \leq s$ for some $s' \in S$. Therefore, s is ordered regular. □

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