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A note on the problem when FS-domains coincide with RB-domains

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Abstract. In this paper, we introduce the notion of super finitely separating functions which gives a characterization of RB-domains. Then we prove that FS-domains and RB-domains are equivalent in some special cases by the following three claims: a dcpo is an RB-domain if and only if there exists an approximate identity for it consisting of super finitely separating functions; a consistent join-semilattice is an FS-domain if and only if it is an RB-domain; an L-domain is an FS-domain if and only if it is an RB-domain. These results are expected to provide useful hints to the open problem of whether FS-domains are identical with RB-domains.

1 Introduction

In [4, 5], A. Jung introduced the notion of FS-domains (that is, finitely separating domains) and proved that the category **FS** of FS-domains is a maximal Cartesian closed full subcategrory of continuous dcpos. Also in [4, 5], it had been shown that the category **RB** of RB-domains (or retracts of algebraic FS-domains) is Cartesian closed, but its maximality is still an

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open question.

A well-known result is that every RB-domain is an FS-domain. Even though much attention has been paid to the question whether each FS-domain is an RB-domain, it is still an open problem [2, 4, 5]. We only make a brief review for the works which are closely related to this problem. In [6], J.D. Lawson proved that the domain of closed formal balls based on a complete metric space is an FS-domain. Meanwhile, it is still unknown whether this domain is an RB-domain. In [7], J.H. Liang and K. Keimel proved that FS-domains and RB-domains are equivalent for L-domains with least elements. In [3], R. Heckmann obtained some characterizations of FS-domains by power domains. In those characterisations, separation by the elements of a finite set is replaced by separation by a continuous non-deterministic function with finite image.

A basic result about RB-domain is that a dcpo is an RB-domain if and only if it has an approximate identity consisting of deflations [4, 5]. Towards the open problem whether each FS-domain is an RB-domain, a natural ideal is to find a deflation over every finitely separating function. Inspired by the idea of R. Heckmann [3], a possible approach for us is to construct a deflation based on the relating finite subset F_δ over every finitely separating function δ.

In this paper, we introduce the notion of super finitely separating functions which is a special case of finitely separating functions. Here, separation by the elements of a finite set is replaced by an order preserving function with finite image. It is shown that a dcpo is an RB-domain if and only if it has an approximate identity consisting of super finitely separating functions, which can be seen as a characterization of RB-domains. Finally, we show that FS-domains always coincide with RB-domains under some special conditions, such as consistent join-semilattices or L-domains (here, the least element is not necessary). Our result may provide useful hints to the open problem mentioned above.

2 FS-domains and RB-domains

A function $f : S \to T$ between dcpos is said to be *Scott continuous* if it sends directed subsets to directed subsets, and preserves sups of directed subsets. We denote all the Scott continuous funcitons from S to T by $[S \to T]$.

Definition 2.1. [2, 4] An *approximate identity* for a dcpo S is a directed subset $\mathcal{D} \subseteq [S \to S]$ satisfying $\sup \mathcal{D} = id_S$, the identity on S.

Definition 2.2. [2, 4] A Scott continuous function $\delta : S \to S$ on a dcpo S is *finitely separating* if there exists a finite set F_δ such that for each $x \in S$, there exists $y \in F_\delta$ such that $\delta(x) \leq y \leq x$.

(1) A dcpo S is called an FS-*domain* if there is an approximate identity for S consisting of finitely separating functions.

(2) An algebraic FS-domain is called a *bifinite domain*.

(3) A dcpo S is called an RB-*domain* if it is isomorphic to the image of some bifinite domain under a Scott continuous projection. That is, an RB-domain is a continuous retract of some bifinite domain.

Lemma 2.3. [2, 4]

(1) *If* $\mathcal{D} \subseteq [S \to S]$ *is an approximate identity for a dcpo* S*, then* $\mathcal{D}' = \{\delta^2 = \delta \circ \delta : \delta \in \mathcal{D}\}$ *is also an approximate identity for* S*.*

(2) *If a Scott continuous function* $\delta : S \to S$ *on a dcpo* S *is finitely separating, then* $\delta(x) \ll x$ *for each* $x \in S$*.*

Lemma 2.4. [1] *A dcpo* S *is an* RB-*domain if and only if there is an approximate identity for* S *consisting of deflations, where a deflation* $f : S \to S$ *is a Scott continuous function with finite image and* $f(x) \leq x$ *holds for each* $x \in S$*.*

Lemma 2.3 indicates that every bifinite domain is an RB-domain and every RB-domain is an FS-domain.

Example 2.5. [2]

(1) All finite posets are bifinite domains, hence RB-domains and FS-domains.

(2) All bounded complete domains are RB-domains, hence FS-domains.

(3) If a dcpo S has an infinite number of minimal elements, then S is not an FS-domain.

Definition 2.6. [7] A dcpo S is an L-*domain* if for every element x of S, the principal ideal $\downarrow x = \{y \in S : y \leq x\}$ is a complete lattice. In this case, we write $\sup_{\downarrow x}$ for the supremum operation in $\downarrow x$.

Lemma 2.7. [7] *In any* L-*domain* S*, if* $x \leq y$ *and* $\phi \neq A \subseteq \downarrow x$*, then* $\sup_{\downarrow x} A = \sup_{\downarrow y} A$*.*

Corollary 2.8. [7] *For each* L*-domain S with the least element, the following statements are equivalent*:

(1) *S is an* FS*-domain.*

(2) *S is an* RB*-domain.*

Each RB-domain is an FS-domain. However, we do not know whether every FS-domain is an RB-domain. For a positive answer, we need to find a deflation above every finitely separating function δ. We notice that in [3], R. Heckmann uses the existing finite separating set: F_δ to give characterizations of FS domains. Therefore, a possible approach for us is to construct a deflation based on the relating F_δ. The first trouble thing is that for each $x \in S$, there may exist more than one element $y \in F_\delta$ such that $\delta(x) \leq y \leq x$. Using the Axiom of Choice, we provide the following lemma to give an equivalent description of finitely separating functions.

Lemma 2.9. *A Scott continuous function $\delta : S \to S$ on a dcpo S is finitely separating if and only if there exists a function $f_\delta : S \to S$ with finite image such that $\delta(x) \leq f_\delta(x) \leq x$ for each $x \in S$.*

Proof. Suppose $\delta : S \to S$ is finitely separating. For each $x \in S$, there exists an element $y_x \in F$ such that $\delta(x) \leq y_x \leq x$. According to the Axiom of Choice, we define a function $f_\delta : S \to S$ by $f_\delta(x) = y_x$ for each $x \in S$. Obviously, $\mathrm{Im}(f_\delta) \subseteq F$ is finite.

Conversely, let $F = \mathrm{Im}(f_\delta)$. It can be checked that $\delta : S \to S$ is finitely separating. □

Remark 2.10. We remind the reader that the function $f_\delta : S \to S$, given in Lemma 2.9, is not necessary to be order preserving. A typical instance is given in Example 3.10.

3 Super finitely separating functions

In this section, we introduce the concept of super finitely separating functions and show that a dcpo S is an RB-domain if and only if S has an approximate identity consisting of super finitely separating functions. Then we show that FS-domains coincide with RB-domains in one of the following cases: (1) consistent join-semilattices; (2) dual of consistent join-semilattices; (3) L-domains.

Definition 3.1. A Scott continuous function $\delta : S \to S$ on a dcpo S is called *super finitely separating* if there exists an order preserving function $f_\delta : S \to S$ with finite image such that $\delta(x) \leq f_\delta(x) \leq x$ for each $x \in S$.

An immediate conclusion is that every deflation is super finitely separating and every super finitely separating function is finitely separating.

Lemma 3.2. *Let S be a domain and $\delta : S \to S$ be a super finitely separating function. Then there exists a Scott continuous function $\theta : S \to S$ with finite image such that $\delta(x) \leq \theta(x) \leq x$ for each $x \in S$.*

Proof. From Definition 3.1, there exists an order preserving function $f_\delta : S \to S$ with finite image such that $\delta(x) \leq f_\delta(x) \leq x$ for each $x \in S$.

Define $\theta : S \to S$ by $\theta(x) = \sup\{f_\delta(y) : y \ll x\}$ for each $x \in S$. Since S is a domain and $f_\delta : S \to S$ is order preserving, $\theta : S \to S$ is well defined. It is easy to see that θ has finite image and it is order preserving. For each $x \in S$, $\delta(x) = \sup\{\delta(y) : y \ll x\} \leq \sup\{f_\delta(y) : y \ll x\} = \theta(x) = \sup\{f_\delta(y) : y \ll x\} \leq \sup\{y : y \ll x\} = x$.

Suppose that D is a directed subset of S. Then $\theta(\sup D) = \sup\{f_\delta(y) : y \ll \sup D\} = \sup\{f_\delta(y) : \exists d \in D \text{ such that } y \ll d\} = \sup_{d \in D} \sup\{f_\delta(y) : y \ll d\} = \sup_{d \in D} \theta(d)$.

Thus $\theta : S \to S$ is Scott continuous. □

Theorem 3.3. *A dcpo S is an* RB*-domain if and only if there is an approximate identity for S consisting of super finitely separating functions.*

Proof. Suppose S is an RB-domain. Since every deflation is a super finitely separating function, there is an approximate identity for S consisting of super finitely separating functions.

Suppose that there exists an approximate identity $\{\delta_i : i \in I\}$ for S, consisting of super finitely separating functions. By Lemma 3.2, for each δ_i, there exists a deflation θ_i such that $\delta_i(x) \leq \theta_i(x) \leq x$ for each $x \in S$. Since $\sup\{\delta_i : i \in I\} = id_S$, we have $\sup\{\theta_i : i \in I\} = id_S$. We have proved that, S is an RB-domain. □

Definition 3.4. A poset P is said to be a *consistent join-semilattice* if each bounded pair in S has a least upper bound. Equivalently, for each $a, b \in S$, if there exists $c \in S$ such that $a \leq c$ and $b \leq c$, then $a \vee b$ exists.

If the dual of P is a consistent join-semilattice, we call it a *dual consistent join-semilattice*.

Remark 3.5. (1) A join-semilattice is always a consistent join-semilattice.

(2) A bounded complete domain D is always a consistent join-semilattice. However, the converse does not hold in general even if D is an FS-domain. In fact, a bounded complete domain must have the least element, which is different from a consistent join-semilattice.

Proposition 3.6. *If a dcpo S is a consistent join-semilattice (or a dual consistent join-semilattice), then each finitely separating function $\delta : S \to S$ is super finitely separating.*

Proof. Since $\delta : S \to S$ is a finitely separating function, there exists a function $f_\delta : S \to S$ with finite $\mathrm{Im}(\delta)$ such that $\delta(x) \leq f_\delta(x) \leq x$ for each $x \in S$, where $\mathrm{Im}(\delta)$ stands for the image of the function δ.

If S is a consistent join-semilattice, we denote $f_\delta^{'}(x) = \sup\{f_\delta(y) : y \leq x\}$ for each $x \in S$. Then the nonempty subset $\{f_\delta(y) : y \leq x\} \subseteq \mathrm{Im}(\delta)$ is finite and $f_\delta(y) \leq y \leq x$ imply that $f_\delta^{'} : S \to S$ is well defined. For each $x \in S$, $f_\delta^{'}(x) = \sup\{f_\delta(y) : y \leq x\} \leq \sup\{y : y \leq x\} = x$ and $f_\delta^{'}(x) \geq f_\delta(x) \geq \delta(x)$. It is easy to see that $f_\delta^{'}(x_1) \leq f_\delta^{'}(x_2)$ for all $x_1, x_2 \in S$ with $x_1 \leq x_2$. Thus δ is a super finitely separating function on S.

In case that S is a dual consistent join-semilattice, just let $f_\delta^{'}(x) = \inf\{f_\delta(y) : y \geq x\}$ for each $x \in S$. We can get the conclusion that δ is a super finitely separating function on S. □

Corollary 3.7. *A consistent join-semilattice* (*or a dual consistent join-semilattice*) *is an* FS*-domain if and only if it is an* RB*-domain.*

Proof. This follows immediately from Lemma 2.4, Theorem 3.3 and Proposition 3.6. □

It is clear that a sup semilattice is a consistent join-semilattice and an inf semilattice is a dual consistent join-semilattice. Then by Corollary 3.7, for a sup semilattice or an inf semilattice, it is an FS-domain if and only it is an RB-domain.

Proposition 3.8. *If S is an* L*-domain, then each finitely separating function $\delta : S \to S$ is super finitely separating.*

Proof. Based on the proof of Proposition 3.6, to prove this proposition, we only need to show the existence of $\inf\{f_\delta(y) : y \geq x\}$ for each $x \in S$.

Since S is an L-domain, every bounded subset of S has the infimum. In particular, $f_\delta(x) \wedge f_\delta(y)$ exists for each pair $x, y \in S$ with $x \leq y$. This can imply that $\inf\{f_\delta(x) \bigwedge f_\delta(y) : x \leq y\}$ exists for each $x \in S$. Observing the sets $\{f_\delta(y) : x \leq y\}$ and $\{f_\delta(x) \bigwedge f_\delta(y) : x \leq y\}$ have the same lower bounds, we can conclude that $\inf\{f_\delta(y) : y \geq x\}$ exists for each $x \in S$. □

Corollary 3.9. *An* L-*domain is an* FS-*domain if and only if it is an* RB-*domain.*

Proof. This follows immediately from Lemma 2.3, Theorem 3.3 and Proposition 3.8. □

The following example shows that a finitely separating function is not necessary super finitely separating.

Example 3.10. Let S be the dcpo as Fig. 1. Then, $\delta : S \to S$ is defined as follows: $\delta(a_i) = b_i$, $\delta(b_i) = d_i$, $\delta(c_i) = d_i$ for each $i \in \mathbb{N}$; $\delta(a) = b$ and maps others to the least element 0.

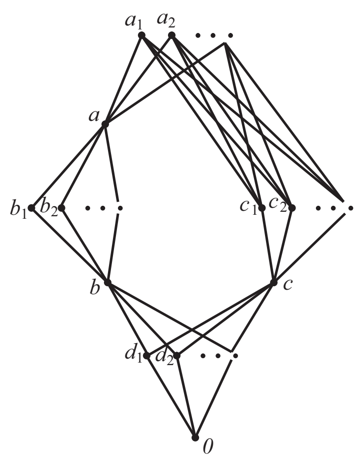

Fig 1

Since every directed subset in S has a maximum element, S is a domain and the order preserving function δ is Scott continuous. It is easy to see that δ is finitely separating if the associated F_δ is chosen as $\{a, b, c, 0\}$. But δ is not super finitely separating. In fact: if a function $f_\delta : S \to S$ with finite image separates δ and id_S , then $f_\delta(a_i) = a$ and $f_\delta(c_i) = c$ hold eventually, but $c \leq a$ is not true, that is to say, f_δ is not order preserving.

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