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Quasi-projective covers of right S-acts

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Abstract. In this paper S is a monoid with a left zero and A_S (or A) is a unitary right S-act. It is shown that a monoid S is right perfect (semiperfect) if and only if every (finitely generated) strongly flat right S-act is quasi-projective. Also it is shown that if every right S-act has a unique zero element, then the existence of a quasi-projective cover for each right act implies that every right act has a projective cover.

1 Introduction and Preliminaries

Let S be a monoid. For right S-acts A and B, A is called B-projective or projective relative to B if for every right S-act C, every homomorphism $f : A \to C$ can be lifted with respect to every epimorphism $g : B \to C$, that is there exists a homomorphism $h : A \to B$ such that $f = gh$. A_S is called projective if it is projective relative to every right S-act. Also A is called *quasi-projective* if A is A-projective and is called *weakly-projective* if A is projective relative to S_S ([1, 7]). There are quite a few papers describing projective acts and their generalizations. Some other generalizations of projectivity are principal weak projectivity, Rees weak projectivity and principal Rees weak projectivity, see [6]. Quasi-projective acts have been studied by Ahsan and Saifullah [1]. Also the concept of weakly-projective acts have been introduced by Knauer and Olthmanns [7]. In this paper we study the concept of quasi-projective cover. Recall that over a monoid S, an S-act A has a projective cover P if there is an epimorphism $f : P \to A$

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such that, P is projective and $f|_C : C \to A$ is not epimorphism for every subact C of P (see [5]). Similar to projective cover (as above) we can define quasi-projective cover, noting that P has to be quasi-projective in this case. Monoids which have a projective cover for each right act are called right perfect monoids. For more details concerning covers of acts, see [2, 3, 4, 8]. In [2], Fountain proved that a monoid S is right perfect if and only if every strongly flat right S-act is projective. From this point of view, we prove that for a monoid S to be right perfect it is enough to show that every strongly flat right S-act is quasi-projective (see Theorem 2.5). Also we give a characterization for monoids for which every cyclic strongly flat act is projective. To give the main result, we focus our attention on right S-acts which have a unique zero. It is shown that if each right S-act has a quasi-projective cover, then S is right perfect.

Modifying the proof of Lemma 1 of [1], we can deduce the following lemma.

Lemma 1.1. ([1]) *Let S be a monoid with a left zero and $\varphi : A_S \to B_S$ be an S-epimorphism. If $A_S \sqcup B_S$ is quasi-projective, then B is a retract of A.*

By the above lemma it is easy to see that over a monoid S with a left zero an S-act (a finitely generated S-act) A_S is projective if and only if there exists an epimorphism $g : P \to A$ such that P is a (finitely generated) projective right S-act and $P_S \sqcup A_S$ is quasi-projective. This fact implies the following theorem:

Theorem 1.2. *Suppose S is a monoid with a left zero and X is a property of acts which is preserved under coproduct and is weaker than projectivity (such as strongly flatness, flatness and etc.), then the following are equivalent:*

(i) *Every (finitely generated) right S-act with property X is quasi-projective.*

(ii) *Every (finitely generated) right S-act with property X is projective.*

By Theorem 4.10.5 of [5] and Theorem 1.2, the following result holds.

Corollary 1.3. *Over a monoid S with a left zero the following are equivalent:*

(i) *Every principally weakly flat right S-act is quasi-projective.*

(ii) *Every weakly flat right S-act is quasi-projective.*

(iii) *Every flat right S-act is quasi-projective.*

(iv) *Every flat right S-act is projective.*

(v) $S = \{1\}$

From Theorem 4.11.8 of [5] and Theorem 1.2, we can deduce the following Corollary.

Corollary 1.4. *Suppose S is a monoid with a left zero, then the following are equivalent:*

(i) *All finitely generated right S-acts which satisfy Condition* (P) *are quasi-projective.*

(ii) *Every right reversible submonoid of S contains a left zero.*

Recall that a right ideal K of a monoid S satisfies Condition (LU) if for every $x \in K$, $x \in Kx$ ([5]).

Proposition 1.5. *Let S be a commutative monoid, then the following are equivalent:*

(i) *All quasi-projective acts over S are flat.*

(ii) *All quasi-projective acts over S are weakly flat.*

(iii) *All quasi-projective acts over S are principally weakly flat.*

(iv) *S is a regular monoid.*

Proof. (i)⇒(ii), (ii)⇒(iii) are obvious.
(iii)⇒(iv). It is easy to see that over commutative monoids every cyclic act is quasi-projective. Thus for every $s \in S$, $\frac{S}{sS}$ is quasi-projective and so is principally weakly flat by assumption. Hence sS satisfies Condition (LU) and so s is regular.
(iv)⇒(i). It is well known that over a commutative regular monoid S, every act is flat. □

2 Semiperfect and perfect monoids with a left zero

Recall that a monoid S is right semiperfect if all cyclic strongly flat right S-acts are projective ([9]). In this section we give a new characterization of semiperfect and perfect monoids with a left zero. We present some results that we need in the sequel.

Proposition 2.1. *Let B_S be an A_S-projective S-act. If C_S is either an S-homomorphic image or an S-subact of A_S, then B_S is C_S-projective.*

Proof. Clearly, if C_S is a homomorphic image of A_S, then the result holds. Thus suppose C_S is a subact of A_S and consider an S-epimorphism $f : C \to \bar{C}$ and an S-homomorphism $g : B \to \bar{C}$ where $\bar{C}$ is a right S-act. Let $\rho = ker f \cup \Delta_A$ where Δ_A is the diagonal relation on A. Clearly $\bar{C} \simeq C/ker f$. Thus if $\pi : A \to A/\rho$ is the natural epimorphism, then π is an extension of f. Since B is A-projective, there exists $h : B \to A$ such that $\pi \circ h = g$. It is easy to see that $h(B) \subseteq C$ and so h is an S-homomorphism from B to C, which proves that B is C-projective. □

One can easily see the following result.

Lemma 2.2. *Suppose S is a monoid and A_S is a right S-act, then:*

(i) *If A is a cyclic right S-act, then A is projective if and only if A is weakly-projective.*

(ii) *If S contains a left zero and $A = \coprod_{i \in I} A_i$ is weakly-projective, then A_i is weakly-projective for every $i \in I$.*

Lemma 2.3. *Suppose S is a monoid with a left zero. If every finitely generated (strongly flat) right S-act has a quasi-projective cover, then every finitely generated (strongly flat) right S-act has a projective cover.*

Proof. By Lemma 1.1 of [4] (Proposition 3.13.14 of [3] and Proposition 1.6 of [4]), it is sufficient to show that every cyclic (strongly flat) right S-act has a projective cover. Let $M = mS$ be a cyclic (strongly flat) right S-act and $\varphi : F \to M$ be an epimorphism such that F is a free S-act. Note that F can be regarded as a cyclic right S-act , because if $F = \coprod_{i \in I} a_i S$ and $m = \varphi(a_j t)$ for some $t \in S$ and $j \in I$, then $\varphi|_{a_j S} : a_j S \to mS$ is an epimorphism. Thus if F is not cyclic we can consider the new epimorphism replace φ. Clearly, $F_S \sqcup M_S$ is finitely generated (strongly flat) and has a quasi-projective cover Q with an epimorphism $\psi : Q \to F_S \sqcup M_S$. Since F is cyclic, Q is finitely generated with two generators. If $F = aS$, then there exist $p, q \in Q$ such that $\psi(p) = m, \psi(q) = a$ and $Q = pS \sqcup qS$. Thus $\pi_F \circ \psi : Q \to F$ is an epimorphism. Since F is projective, there exists a homomorphism $h : F \to Q$ such that $\pi_F \circ \psi \circ h = 1_F$ and hence h is a coretraction. Since $F_S \simeq S_S$ and h is a monomorphism, S_S is a subact of Q. Thus by Proposition 2.1, Q is weakly-projective and by Lemma 2.2(i), it is projective. Clearly pS is the projective cover of M. □

By the following theorem, we show that for a monoid S with a left zero to be semiperfect it is enough to show that every finitely generated strongly flat right S-act has a quasi-projective cover.

Theorem 2.4. *For a monoid S with a left zero the following are equivalent:*

(i) *S is right semiperfect.*

(ii) *Every finitely generated strongly flat right S-act has a quasi-projective cover.*

(iii) *Every finitely generated strongly flat right S-act has a weakly-projective cover.*

(iv) *Every cyclic strongly flat right S-act has a weakly-projective cover.*

(v) *Every left collapsible submonoid of S contains a left zero (Condition* (K)*).*

Proof. (i)⇒(ii), (i)⇒(iii). By Proposition 3.13.14 of [5] are clear. (iii)⇒(iv) is clear. (ii)⇒(i). By Lemma 2.3, every finitely generated strongly flat right S-act A_S, has a projective cover and so it is projective by Proposition 1.7 of [4]. (iv)⇒(i). If $A = aS$ is a strongly flat right S-act, then every cover of A is cyclic. Now the result follows by Proposition 1.7 of [4] and Lemma 2.2(i). The equivalence of (i) and (v) follows by Theorem 4.11.2 of [5]. □

Recall that a monoid S satisfies Condition(A) if every right S-act satisfies the ascending chain condition for its cyclic subacts ([5]). Fountain in [2], proved that a monoid S is right perfect if and only if every strongly flat right S-act is projective. The next theorem improves this result by the notion of quasi-projectivity.

Theorem 2.5. *Let S be a monoid with a left zero. The following are equivalent:*

(i) *S is right perfect.*

(ii) *Every strongly flat right S-act is quasi-projective.*

(iii) *S satisfies Condition* (A) *and every finitely generated strongly flat right S-act has a quasi-projective cover.*

(iv) *S satisfies Condition* (A) *and every cyclic strongly flat right S-act has a weakly-projective cover.*

Proof. (i)⇒(ii) is clear. (ii)⇒(i). Suppose every strongly flat right S-act is quasi-projective. Then by Theorem 1.2, every strongly flat right S-act is projective. Thus S is right perfect by Theorem 1.8 of [4]. The equivalences of (i) and (iii), and also (i) and (iv) follow by Theorem 4.11.6 of [5] and Theorem 2.4. □

Now we state the main result.

Theorem 2.6. *Suppose S is a monoid with a left zero and every right S-act has only one zero element. If every right S-act has a quasi-projective cover, then S is right perfect.*

Proof. We show that every right S-act has a projective cover. Suppose M_S is a right S-act and $\phi : F \to M$ is an epimorphism such that F_S is a free S-act. Let $F' = F - \{\theta_F\}$ and $M' = M - \{\theta_M\}$ and $B = F' \sqcup M' \sqcup \theta$, where θ is the one-element right S-act. Then B is a right S-act by the right S-action, $\theta.s = \theta$ and

$$b.s = \begin{cases} \theta, & \text{if } bs = \theta_F \textit{ or } \theta_M; \\ bs, & \textit{otherwise} \end{cases} \tag{1}$$

for every $s \in S$ and $b \in F' \sqcup M'$.
Suppose Q is a quasi-projective cover of $F' \sqcup M' \sqcup \theta$ with an epimorphism

$\pi : Q \to F' \sqcup M' \sqcup \theta$. Now define $q : F' \sqcup M' \sqcup \theta \to F' \sqcup \theta$ by

$$q(x) = \begin{cases} x, & x \in F' \sqcup \theta; \\ \theta, & x \in M'. \end{cases} \tag{2}$$

Clearly q is a homomorphism. Now consider the following diagram:

$$\begin{array}{ccccc} & & & & F' \sqcup \theta \\ & & & & \downarrow 1_{F' \sqcup \theta} \\ Q & \xrightarrow{\pi} & F' \sqcup M' & \xrightarrow{q} & F' \sqcup \theta \end{array}$$

Since $F' \sqcup \theta \simeq F$ is projective, there exists $i : F' \sqcup \theta \to Q$ such that $q \circ \pi \circ i = 1_{F' \sqcup \theta}$. Thus i is a monomorphism and we can regard F as a subact of Q. Let $K = \{x \in Q : q(\pi(x)) = \theta_F\}$ and $K' = K - \{\theta_Q\}$. Clearly $K' \sqcup i(F' \sqcup \theta)$ is a subact of Q. We show that $\pi_1 = \pi|_{K' \sqcup i(F' \sqcup \theta)} : K' \sqcup i(F' \sqcup \theta) \to F' \sqcup M' \sqcup \theta$ is an epimorphism. For this we show that $\pi(i(x)) = x$, for every $x \in F' \sqcup \theta$. Suppose $x \in F' \sqcup \theta$. If $x = \theta$, then clearly $\pi(i(\theta)) = \theta$. Suppose $x \in F'$ and let $z = \pi(i(x))$. Then $q(z) = q(\pi(i(x))) = x$. Thus $q(z) = x \in F'$. By the definition of q, $q(z) = z$, i.e., $z = x$. Thus $\pi(i(x)) = x$ for every $x \in F' \sqcup \theta$. Thus π_1 is an epimorphism and since π is coessential, $Q = K' \sqcup i(F' \sqcup \theta) \simeq K' \sqcup F' \sqcup \theta$. Now let $\pi_2 = \pi|_{K' \sqcup \theta} : (K' \sqcup \theta) \simeq K \to (M' \sqcup \theta) \simeq M$. Since π is coessential π_2 is a coessential epimorphism. Since F is a projective S-act, there exists $\phi' : F \to K$ such that the diagram

$$\begin{array}{ccc} & & F \\ & \phi' \swarrow & \downarrow Q \\ K & \xrightarrow{\pi_2} & M \end{array}$$

is commutative and $\pi_2 \circ \phi' = \phi$. Thus $\pi_2(\phi'(F)) = \phi(F) = M$ and, since π_2 is coessential, ϕ' is an epimorphism. Now define
$q' : F' \sqcup K' \sqcup \theta \to K' \sqcup \theta$ by

$$q'(x) = \begin{cases} x, & x \in K' \sqcup \theta; \\ \theta, & x \in F', \end{cases} \tag{3}$$

and $q'' : F' \sqcup K' \sqcup \theta \to F' \sqcup \theta$ by

$$q''(x) = \begin{cases} x, & x \in F' \sqcup \theta; \\ \theta, & x \in K'. \end{cases} \tag{4}$$

Clearly q' and q'' are homomorphism. Now consider the following diagram

$$\begin{array}{ccccc} & & & & F' \sqcup K' \sqcup \theta \\ & & & & \big\downarrow{\scriptstyle q'} \\ & & & & K' \sqcup \theta \\ & & & & \big\downarrow{\scriptstyle 1_{K' \sqcup \theta}} \\ F' \sqcup K' \sqcup \theta & \xrightarrow{q''} & (F' \sqcup \theta) \simeq F & \xrightarrow{\phi'} & (K' \sqcup \theta) \simeq K \end{array}$$

Since $F' \sqcup K' \sqcup \theta \simeq Q$ is quasi-projective, there exists $h : F' \sqcup K' \sqcup \theta \to F' \sqcup K' \sqcup \theta$ such that $\phi' \circ q'' \circ h = 1_{K' \sqcup \theta} \circ q'$. If $j : K' \sqcup \theta \to F' \sqcup K' \sqcup \theta$ is the canonical injection, then $q' \circ j = 1_{K' \sqcup \theta}$ and so $\phi' \circ q'' \circ h \circ j = 1_{K' \sqcup \theta}$. Thus $K \simeq K' \sqcup \theta$ is a retract of $F' \sqcup \theta \simeq F$ and so is projective. Hence K is the projective cover of M. □

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